

Connectivity & electricity

Lecture 16

Probability (and techniques)

Algorithms (models and applications)

Random Walks

Stationary Distribution

Perron-Frobenius Theorem

Pagerank

connectivity

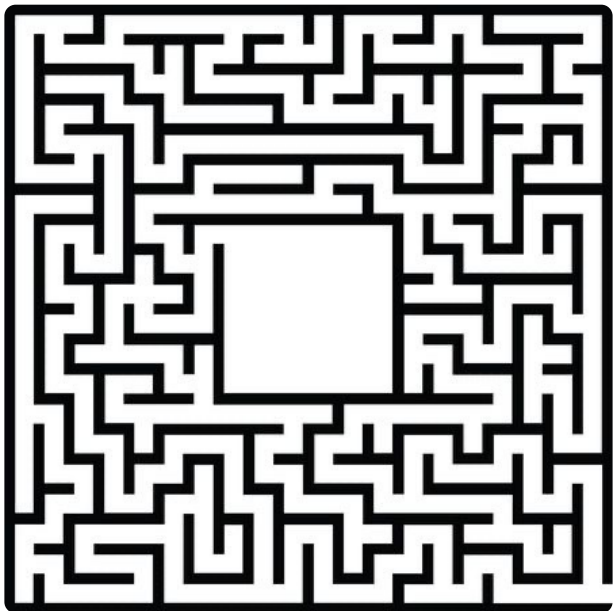
Input: $G=(V,E)$ undirected,
(m edges, n vertices)

$s, t \in V$

Q: is there a path from
s to t?

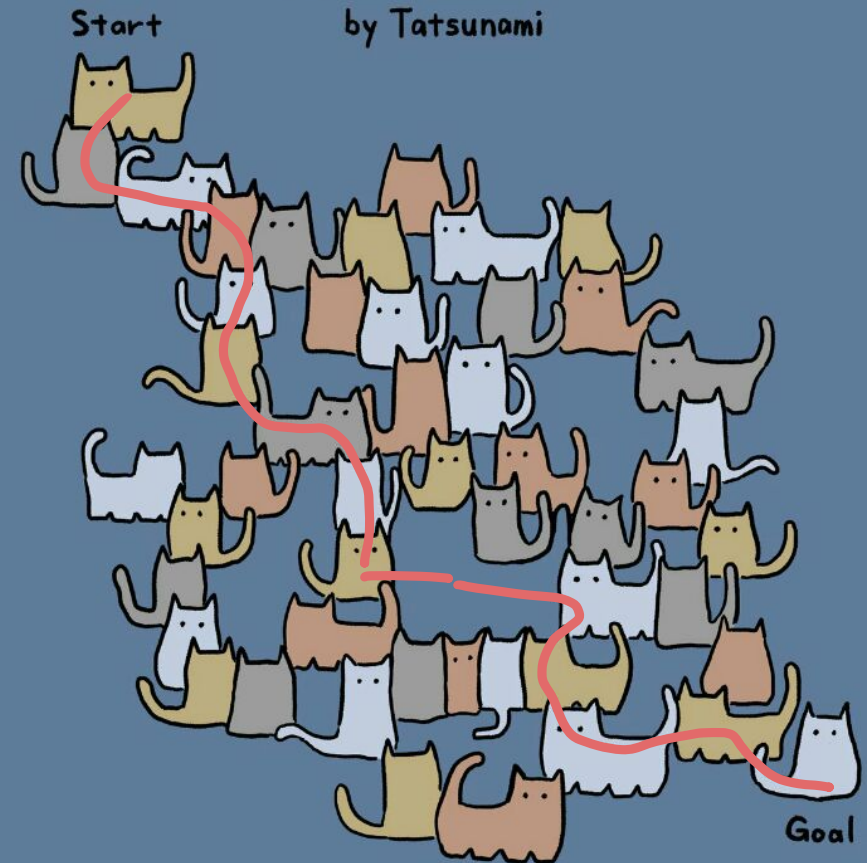
3 links

Yes	No	Not voting
5	18	6



CAT MAZE

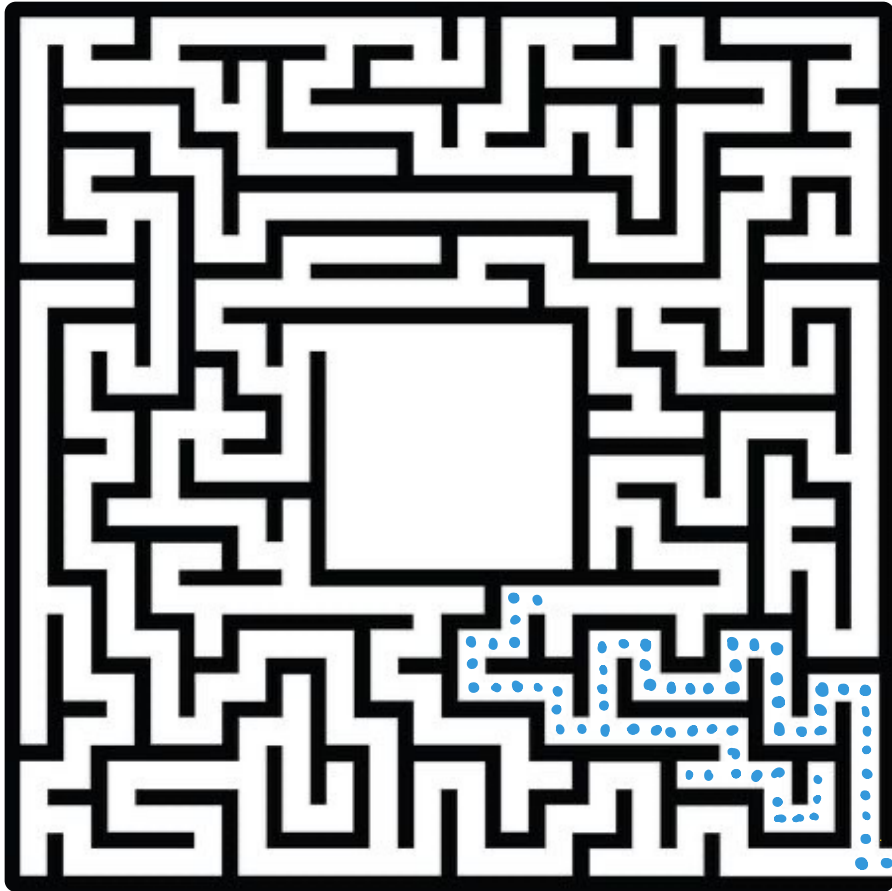
by Tatsunami



Reach the Goal by following the cats, alternating between cats facing the front and cats facing backwards.

The cats in your path must be connected or overlapping.

It doesn't matter what color or size they are.



Breadcrumbs

BFS, DFS take
linear time and space

can connectivity be solved in...

	Yes	No	Not voting
linear space?	96%		4% / 1 vote
$O(\log^{O(1)} n)$ space?	96%	1	
$O(\log n)$ space?	91%	5	
$O(\log n)$ space deterministically?	7	92.5%	

can connectivity be solved in...

Yes

No

Not
voting

linear space?

BFS,
DFS

$O(\log^{O(1)} n)$ space?

Savitch

$O(\log n)$ space?

today

$O(\log n)$ space
deterministically?

in a
few
lectures!

Lecture 18

Probability
(and techniques)

Hitting times

Cover times

Electrical networks

Algorithms
(models and applications)

Connectivity
in $O(\log n)$ space

can connectivity be solved in $O(\log n)$ space?

algorithm:

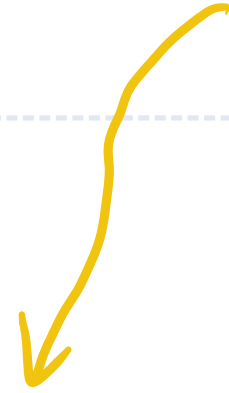
(1) repeat $O(\log n)$ times:

(a) start at $s \in V$

(b) randomly walk for $O(mn)$ steps.

(c) if walk visits t , return TRUE

(2) return False



random walk terminology:

Hitting time $H(s,t)$ = expected # steps until a random walk from s reaches t

Cover time $C(s)$ = expected # steps until a random walk from s visits every vertex

can connectivity be solved in $O(\log n)$ space?

algorithm:

- (1) repeat $O(\log n)$ times:
 - (a) start at $s \in V$
 - (b) randomly walk for $O(mn)$ steps.
 - (c) if walk visits t , return TRUE
- (2) return False

random walk terminology:

Hitting time $H(s, t)$ = expected # steps until a random walk from s reaches t

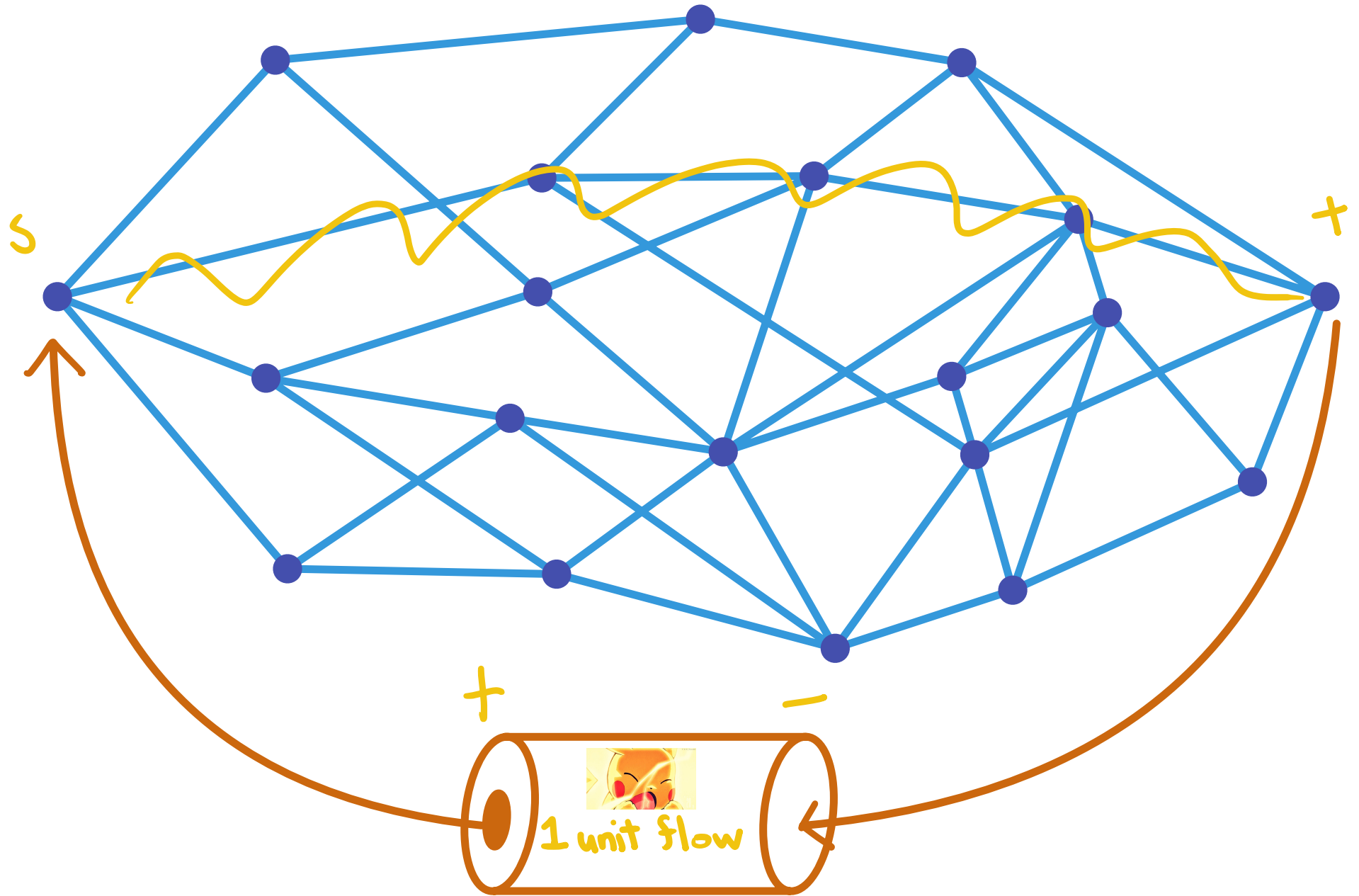
Cover time $C(s)$ = expected # steps until a random walk from s visits every vertex

Theorem

If G is connected, then $C(s) \leq m(n-1)$



Electrical networks



electricity takes the "path of least resistance"

"flow of least resistance"

each edge e has resistance $r_e > 0$

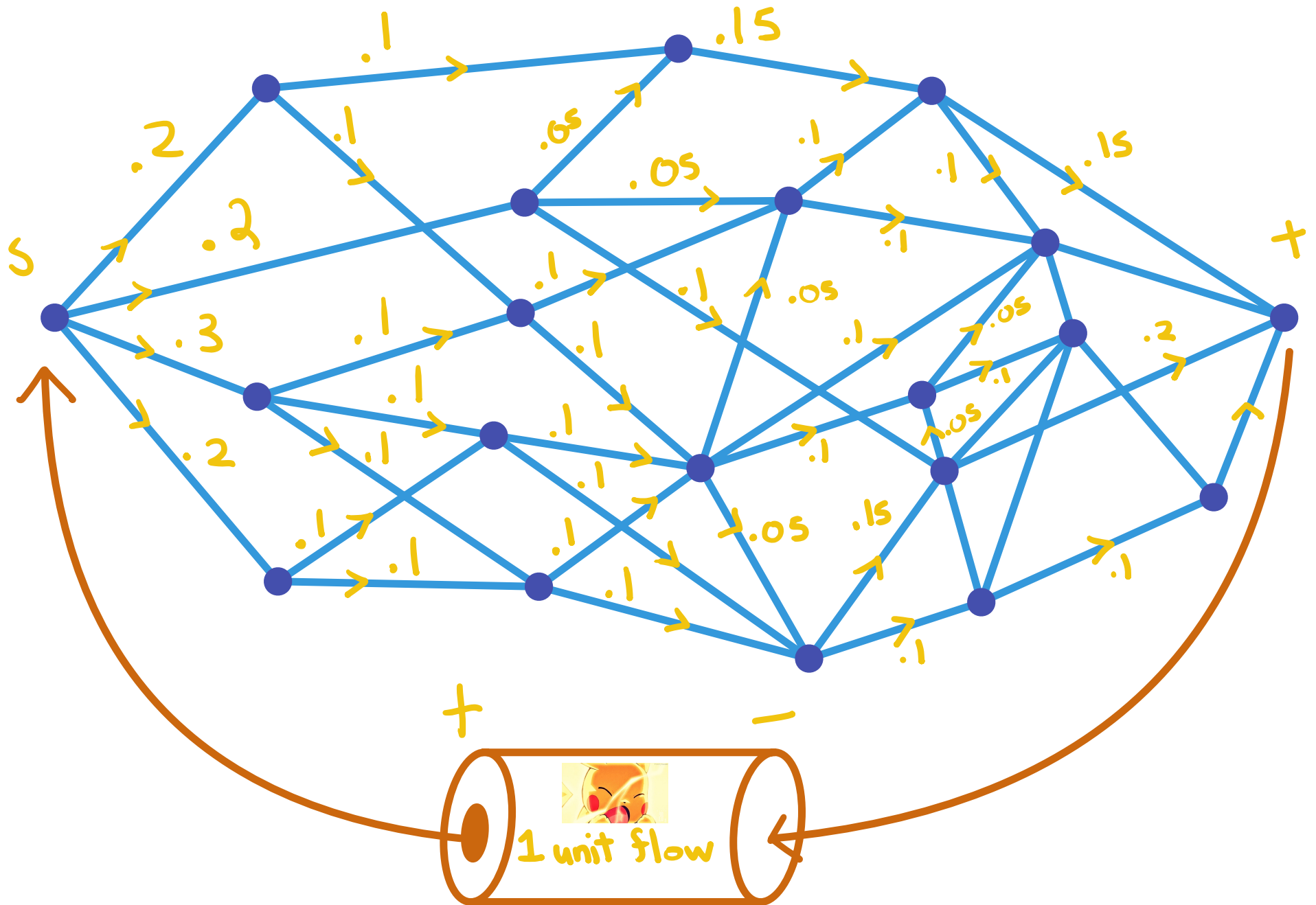
for flow $f: E \rightarrow \mathbb{R}_{\geq 0}$,

$$\text{electrical energy} = \sum_{e \in E} r_e f_e^2$$

electrical flow = flow of least energy

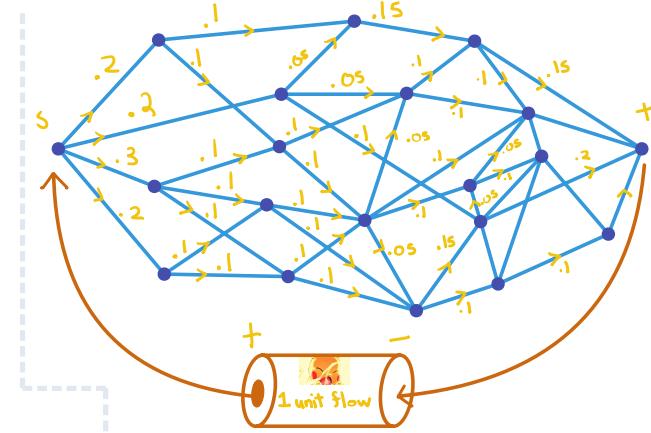
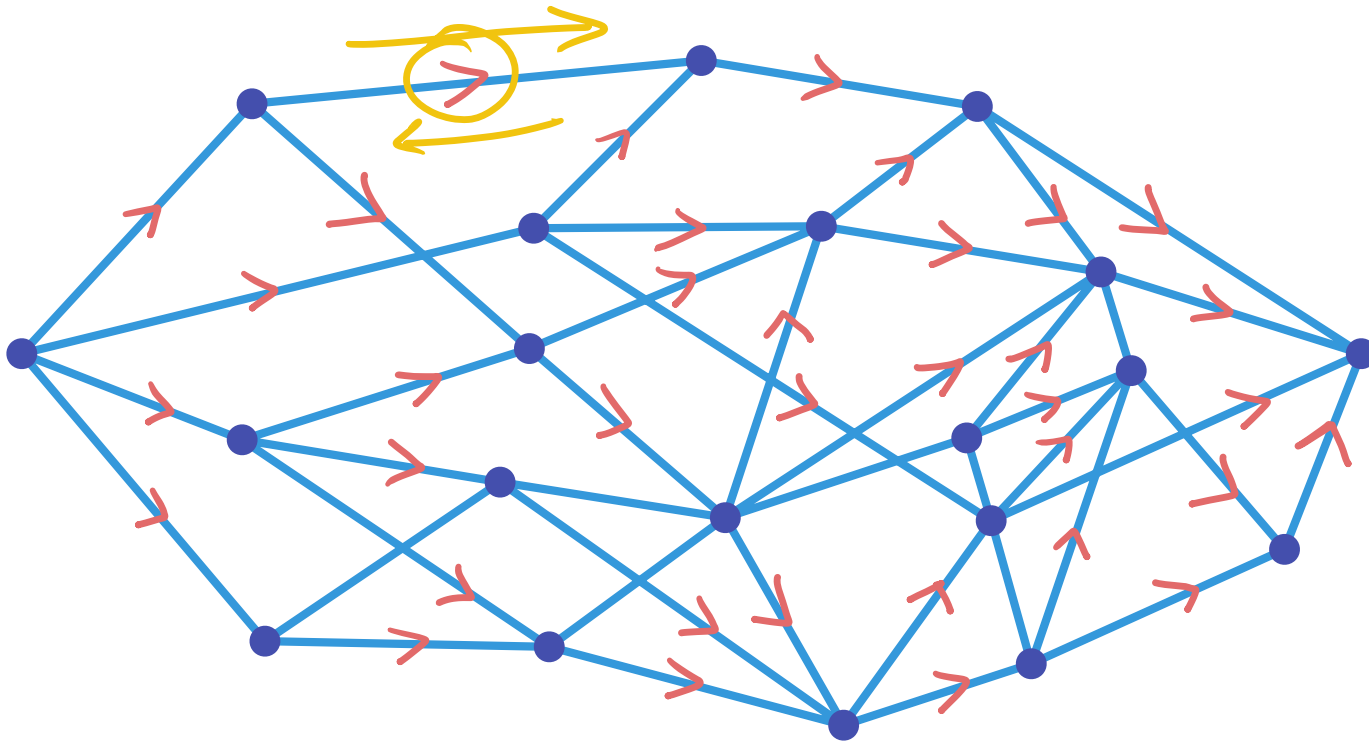
electrical energy = $\frac{1}{2} \sum_{e \in E} r_e f_e^2$

electrical flow = flow of least energy



Electrical flow = convex minimization
math setup

orient edges arbitrarily:



$$\text{electrical energy} = \sum_{e \in E} r_e f_e^2$$

electrical flow = flow of least energy

$f_e > 0 \Rightarrow |f_e|$ units flow in direction of e

$f_e < 0 \Rightarrow |f_e|$ units flow in opposite direction

Electrical flow = convex minimization
math setup

Flow constraints = linear system.

def $B: \mathbb{R}^E \rightarrow \mathbb{R}^V$ by

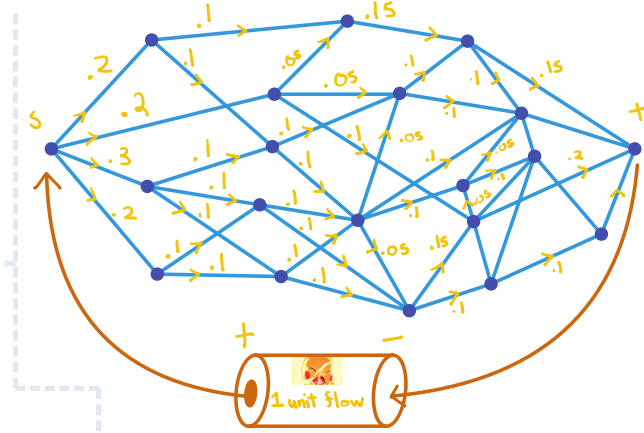
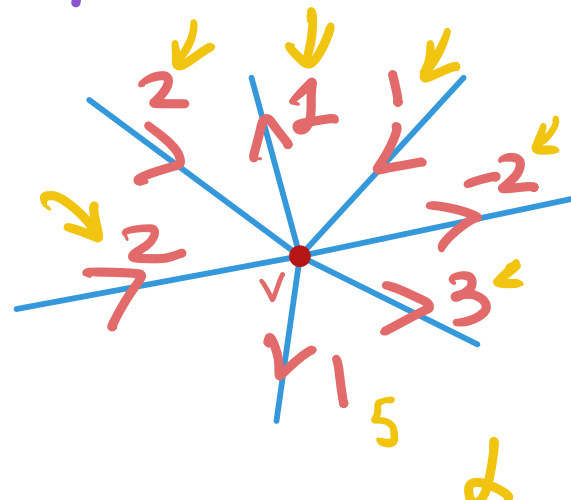
$$(Bf)_v = \text{net flow at } v$$

B is a linear function (check!)

$\Rightarrow B$ is a matrix ($\mathbb{R}^{V \times E}$)

we want

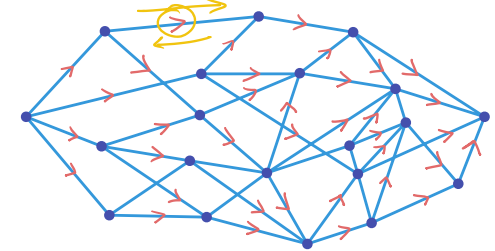
$$Bf = d \quad \text{where} \quad d_v = \begin{cases} +1 & \text{if } v=s \\ -1 & \text{if } v=t \\ 0 & \text{otherwise} \end{cases}$$



$$\text{electrical energy} = \sum_{e \in E} r_e f_e^2$$

electrical flow = flow of least energy

orient edges arbitrarily:



$f_e > 0 \Rightarrow |f_e|$ units flow in direction of e

$f_e < 0 \Rightarrow |f_e|$ units flow in opposite direction

Electrical flow = convex minimization

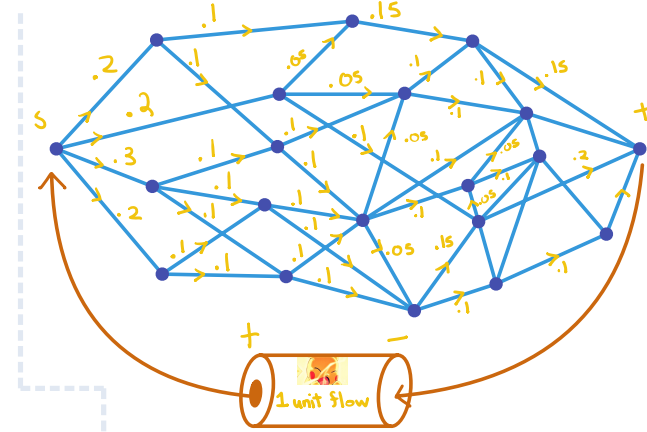
$$\min_{f \in \mathbb{R}^E} \quad \frac{1}{2} \sum_e r_e f_e^2$$

(electrical energy)

s.t. $Bf = d$
(f routes d)

electrical flow = optimum solution

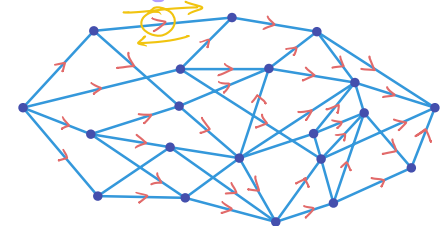
$\Rightarrow$ we should study (general)
optimality conditions



$$\text{electrical energy} = \sum_{e \in E} r_e f_e^2$$

electrical flow = flow of least energy

orient edges arbitrarily:



$f_e > 0 \Rightarrow |f_e|$ units slow in direction of e

$f_e < 0 \Rightarrow |f_e|$ units flow in opposite direction

def B: $\mathbb{R}^E \rightarrow \mathbb{R}^V$ by

$$(BF)_v = \text{net flow at } v$$

B is a linear function (check!)

$\Rightarrow B$ is a matrix ($\mathbb{R}^{V \times E}$)

we want $B\xi = d$

where $d_v = \begin{cases} +1 & \text{if } v=s \\ -1 & \text{if } v=t \\ 0 & \text{otherwise} \end{cases}$

$$\min_{\mathbf{f} \in \mathbb{R}^E} \quad \frac{1}{2} \sum_e r_e f_e^2$$

(electrical energy)

s.t. $Bf = d$
(f routes d)

(First order conditions)

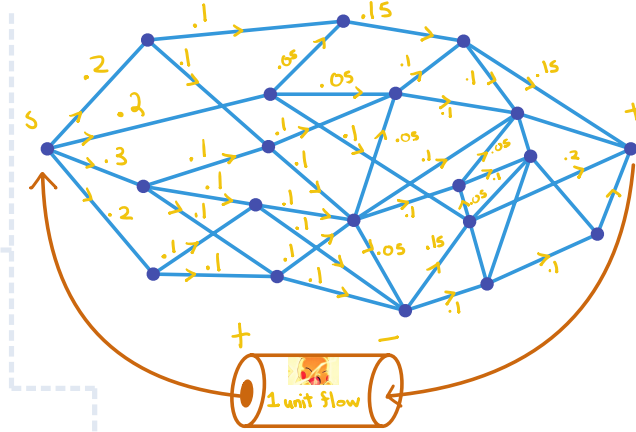
"Ohm's Law" let f route d.

f is the electrical flow

$$\Leftrightarrow f_e = \frac{p_v - p_u}{r_e} \quad \forall e = (u, v)$$

for some $p: V \rightarrow \mathbb{R}$

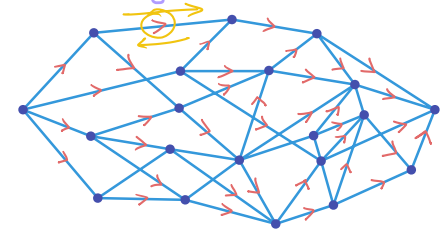
"vertex potentials"



$$\text{electrical energy} = \sum_{e \in E} r_e f_e^2$$

electrical flow = flow of least energy

orient edges arbitrarily:



$f_e > 0 \Rightarrow |f_e|$ units flow in direction of e

$f_e < 0 \Rightarrow |f_e|$ units flow in opposite direction

des $B: \mathbb{R}^E \rightarrow \mathbb{R}^V$ by
(flows)

$$(BF)_v = \text{net flow at } v$$

B is a linear function (check!)

$\Rightarrow B$ is a matrix $(\mathbb{R}^{V \times E})$

we want $Bf = d$

where $d_v = \begin{cases} +1 & \text{if } v=s \\ -1 & \text{if } v=t \\ 0 & \text{otherwise} \end{cases}$

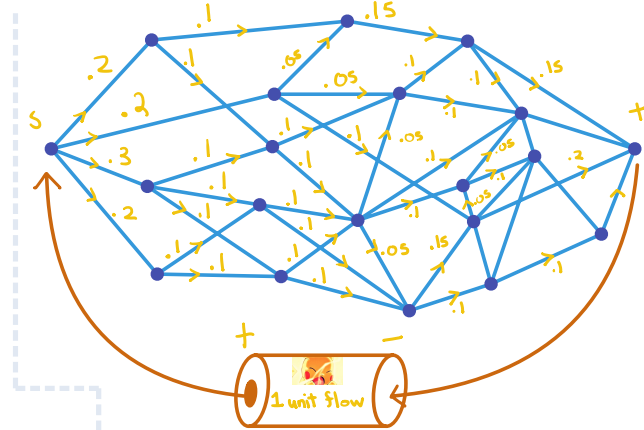
(First order conditions)

"Ohm's Law" let f route d .

f is the electrical flow

$$\Leftrightarrow f_e = \frac{p_v - p_u}{r_e} \quad \forall e = (u, v)$$

"vertex potentials"
for some $p: V \rightarrow \mathbb{R}$



$$\text{electrical energy} = \sum_{e \in E} r_e f_e^2$$

electrical flow = flow of least energy

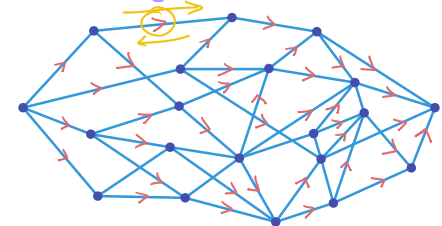
Proof ($\Rightarrow$) let f be electrical flow

edge lengths $l(e) = r_e f_e$ (= gradient)

claim: no negative cycles w/r/t l



orient edges arbitrarily:



$f_e > 0 \Rightarrow |f_e|$ units flow in direction of e

$f_e < 0 \Rightarrow |f_e|$ units flow in opposite direction

def $B: \mathbb{R}^E \rightarrow \mathbb{R}^V$ by

$(Bf)_v = \text{net flow at } v$

B is a linear function (check!)

$\Rightarrow B$ is a matrix $(\mathbb{R}^{V \times E})$

we want $Bf = d$

where $d_v = \begin{cases} +1 & \text{if } v = s \\ -1 & \text{if } v = t \\ 0 & \text{otherwise} \end{cases}$

(First order conditions)

"Ohm's Law"

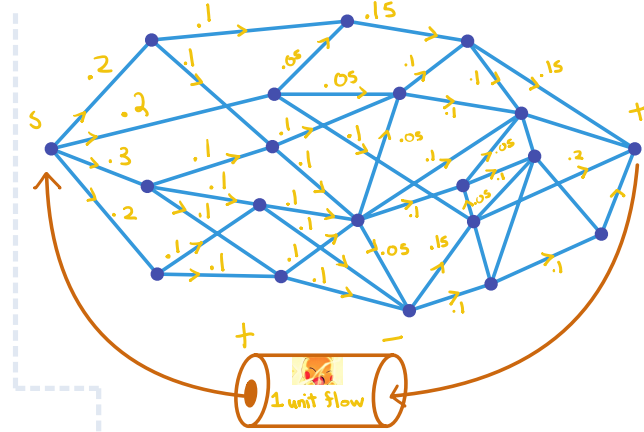
f is the electrical flow

let f route d.

$$p_v - p_u = l(e)$$

$$\Leftrightarrow f_e = \frac{p_v - p_u}{r_e} \quad \forall e = (u, v)$$

"vertex potentials"
for some $p: V \rightarrow \mathbb{R}$



Proof ($\Rightarrow$) let f be electrical flow

$$\text{edge lengths } l(e) = r_e f_e \quad (= \text{gradient})$$

$$\text{electrical energy} = \sum_{e \in E} r_e f_e^2$$

electrical flow = flow of least energy

claim: no negative cycles w/r/t l $e = (u, v)$

def.

$$\text{potentials } p(v) = \min_s d_l(s, v) \quad (\text{shortest path distance w/r/t } l)$$

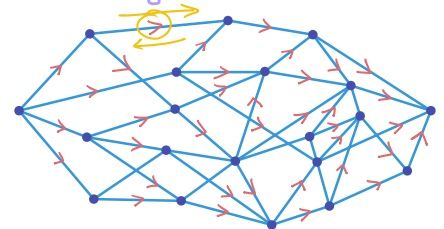
$$p(v) - p(u) \leq l(u, v) \quad (\text{triangle inequality})$$

$$p(u) - p(v) \leq l(v, u) \quad (\text{triangle inequality})$$

$$p_v - p_u = r_e f_e$$
$$e = (u, v)$$

$$l(u, v) = -l(v, u) = r_{uv} f(u, v) = -r_{uv} f(v, u)$$

orient edges arbitrarily:



$f_e > 0 \Rightarrow |f_e|$ units flow in direction of e

$f_e < 0 \Rightarrow |f_e|$ units flow in opposite direction

def $B: \mathbb{R}^E \rightarrow \mathbb{R}^V$ by
(flows)

$(Bf)_v = \text{net flow at } v$
 B is a linear function (check!)
 $\Rightarrow B$ is a matrix $(\mathbb{R}^{V \times E})$

we want $Bf = d$

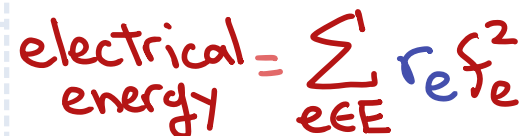
where $d_v = \begin{cases} +1 & \text{if } v = s \\ -1 & \text{if } v = t \\ 0 & \text{otherwise} \end{cases}$

"Ohm's Law" let f route d.

$$\Leftrightarrow f_e = \frac{p_v - p_u}{r_e} \quad \forall e = (u, v)$$

"vertex potentials"

for some $p: V \rightarrow \mathbb{R}$



electrical flow = flow of least energy

A diagram of a 2D hexagonal lattice. The lattice is composed of blue lines (bonds) connecting black dots (sites). Red arrows are placed on the bonds, pointing in various directions, representing a spin configuration. A yellow circle with a double-headed arrow is shown above the lattice, indicating a magnetic field or a specific spin state.

$f_e < 0 \Rightarrow |f_e|$ units flow in opposite direction

$$(Bf)_v = \text{net flow at } v$$

$\Rightarrow B$ is a matrix $(\mathbb{R}^{V \times E})$

where $d_v = \begin{cases} +1 & \text{if } v=s \\ -1 & \text{if } v=t \\ 0 & \text{otherwise} \end{cases}$

Proof ($\Leftarrow$) Suppose

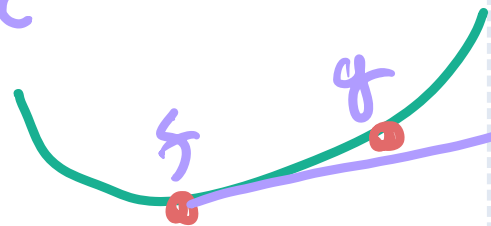
$g =$ any flow routing d

$f - g$ is a circulation (net flow 0 everywhere)
(sum of cycles)

edge lengths $l(e) = r_e f_e = p_v - p_u \quad \forall e = (u, v)$

cycle C : $l(C) = \sum_{u,v \in C} l(u,v) = \sum_{u,v \in C} p_v - p_u = 0$

Convexity of $\mathcal{E}(f) = \sum_e r_e f_e^2$:

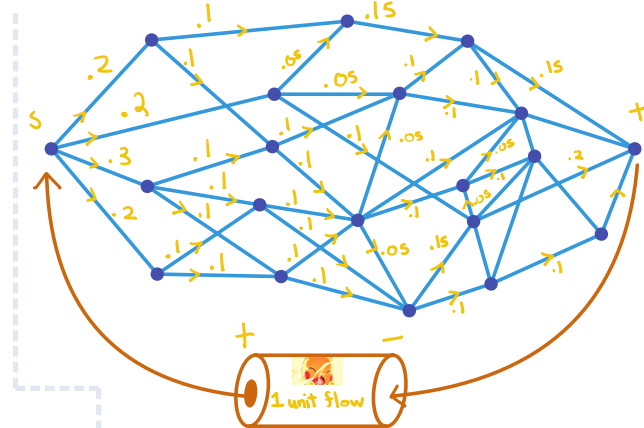


$$\mathcal{E}(g) \geq \mathcal{E}(f) + \langle \mathcal{E}'(f), g-f \rangle$$

$$= \mathcal{E}(f) + \langle \ell, g-f \rangle$$

circulation

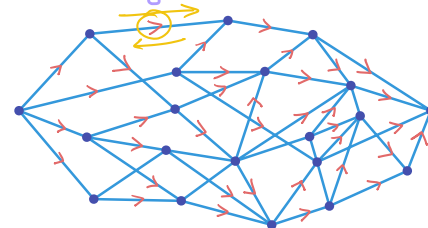
$$= \mathcal{E}(f)$$



$$\text{electrical energy} = \sum_{e \in E} r_e f_e^2$$

electrical flow = least energy

orient edges arbitrarily:



$f_e > 0 \Rightarrow |f_e|$ units flow in direction of e

$f_e < 0 \Rightarrow |f_e|$ units flow in opposite direction

def $B: \mathbb{R}^E \rightarrow \mathbb{R}^V$ by

$(Bf)_v = \text{net flow at } v$

B is a linear function (check!)

$\Rightarrow B$ is a matrix $(\mathbb{R}^{V \times E})$

we want $Bf = d$

where $d_v = \begin{cases} +1 & \text{if } v=s \\ -1 & \text{if } v=t \\ 0 & \text{otherwise} \end{cases}$

Electrical flow = convex minimization

$$\min_{f \in \mathbb{R}^E} \quad \frac{1}{2} \sum_e r_e f_e^2$$

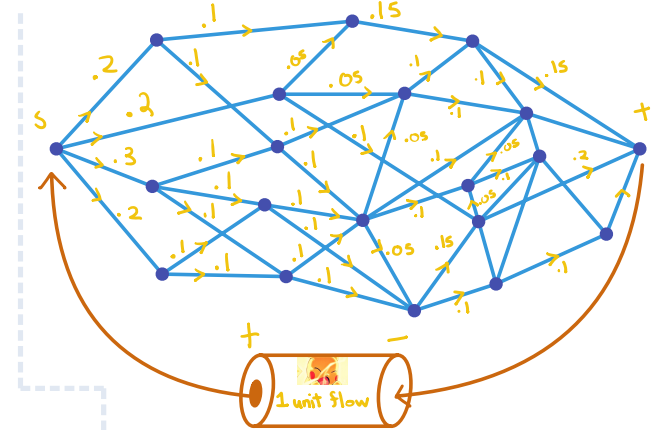
(electrical energy)

s.t. $Bf = d$

$(f \text{ is } (s,t)\text{-flow})$

electrical flow = optimum solution

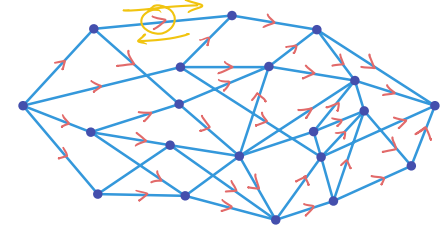
"effective" resistance of d = optimum value/energy



$$\text{electrical energy} = \sum_{e \in E} r_e f_e^2$$

electrical flow = flow of least energy

orient edges arbitrarily:



$f_e > 0 \Rightarrow |f_e|$ units slow in direction of e

$f_e < 0 \Rightarrow |f_e|$ units flow in opposite direction

def $B: \mathbb{R}^E \rightarrow \mathbb{R}^V$ by

$$(Bf)_v = \text{net flow at } v$$

B is a linear function (check!)

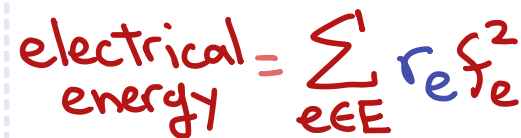
$\Rightarrow B$ is a matrix ($\mathbb{R}^{V \times E}$)

we want $Bf = d$

where $d_v = \begin{cases} +1 & \text{if } v=s \\ -1 & \text{if } v=t \\ 0 & \text{otherwise} \end{cases}$

Laplacian $[L]^{E \times E}$ $R = \text{diag}(r)$

$$R = \text{diag}(r)$$



electrical flow = flow of least energy

$$d_v = \begin{cases} +1 & \text{if } v=s \\ -1 & \text{if } v=t \\ 0 & \text{otherwise} \end{cases}$$

$$\min_{\mathbf{f} \in \mathbb{R}^E} \quad \frac{1}{2} \sum_e r_e f_e^2 \quad \text{s.t.} \quad \mathbf{B}\mathbf{f} = \mathbf{d}$$

(electrical energy) (\$f\$ is (\$s\$)-(\$l\$)-flow)

effective resistance of d = optimum value/energy

Ohm's law: $\xi = R^{-1} B^T p$
 for electric flow ξ , electric potentials p

e.g. $x \in \mathbb{R}^N$

$$\langle x, Lx \rangle = \langle B^T x, R^{-1} B^T x \rangle$$

$$= \sum_{e \in E} \frac{1}{r_e} (B^T x)_e^2$$

$$= \sum_{e=\{u,v\}} \frac{1}{r_e} (x_u - x_v)^2$$

Laplacian

$$L = BR^{-1}B^T \in \mathbb{R}^{V \times V}$$

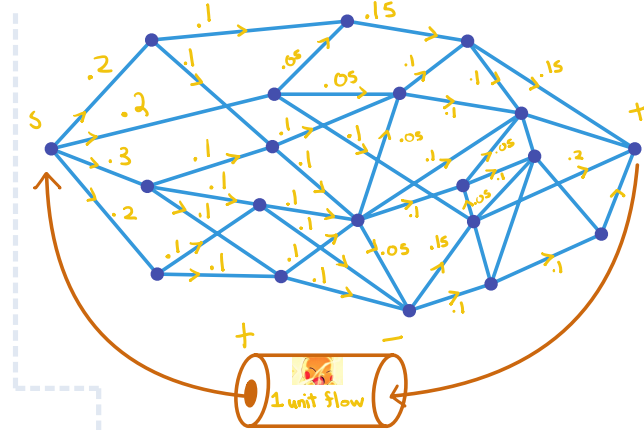
"Laplacian" of graph w/ edge weights $w(e) = \frac{1}{r(e)}$

$$\checkmark \langle x, Lx \rangle = \sum_{e=\{u,v\}} \frac{1}{r_e} (x_u - x_v)^2$$

for $S \subseteq V$, let $x = \frac{1}{2}(\mathbb{1}_S - \mathbb{1}_{\bar{S}})$

$$\langle x, Lx \rangle = \sum_{\substack{e \in \partial(S) \\ e = \{u, v\}}} \frac{1}{r_e} \left(\frac{1}{2} - \left(-\frac{1}{2} \right) \right)^2$$

$$= \sum_{e \in \partial(S)} \frac{1}{r_e} = w(\partial(S))$$



$$\text{electrical energy} = \sum_{e \in E} r_e f_e^2$$

electrical flow = flow of least energy

def $B: \mathbb{R}^E \rightarrow \mathbb{R}^V$ by
 $(Bf)_v = \text{net flow at } v$

$$d_v = \begin{cases} +1 & \text{if } v = s \\ -1 & \text{if } v = t \\ 0 & \text{otherwise} \end{cases}$$

$$\min_{\mathbf{f} \in \mathbb{R}^E} \quad \frac{1}{2} \sum_e r_e f_e^2 \quad \text{s.t.} \quad \mathbf{B}\mathbf{f} = \mathbf{d}$$

(electrical energy) (f is (s)ti-slow)

effective resistance of d = optimum value/energy

Ohm's law: $\xi = R^{-1} B^T p$
 for electric flow ξ , electric potentials p

Lemma $Lp = d$, and

$$(\text{effective resist. of } d) = \langle d, L^{-1}d \rangle = \langle p, d \rangle$$

$$r_e f_e = p_v - p_u$$

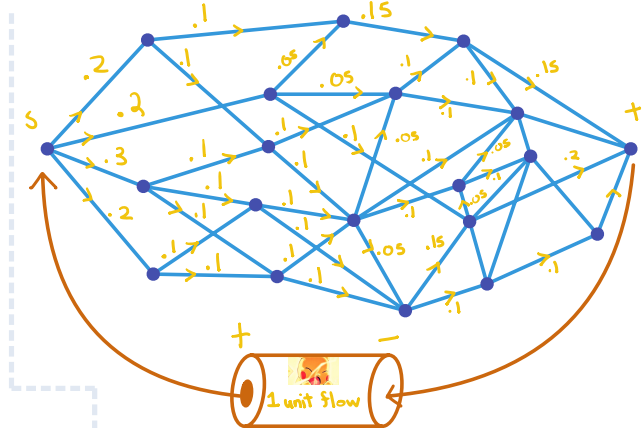
$$Rf = B^T p$$

$$Lp = BR^{-1}B^T p = \cancel{BR^{-1}} Rf = d$$

$$p = L^{-1}d$$

$$\begin{aligned} \langle f, Rf \rangle &= \langle R^{-1}B^T p, \cancel{R} R^{-1} B^T p \rangle \\ &= \langle p, Lp \rangle = \langle Lp, L^{-1}Lp \rangle \\ &= \langle d, L^{-1}d \rangle \end{aligned}$$

electric potentials for d



$$\text{electrical energy} = \sum_{e \in E} r_e f_e^2$$

electrical flow = least energy

def $B: \mathbb{R}^E \rightarrow \mathbb{R}^V$ by
 $(Bf)_v = \text{net flow at } v$
 $d_v = \begin{cases} +1 & \text{if } v=s \\ -1 & \text{if } v=t \\ 0 & \text{otherwise} \end{cases}$

$$\min_{f \in \mathbb{R}^E} \frac{1}{2} \sum_e r_e f_e^2 \text{ s.t. } Bf = d$$

(electrical energy) (f is (s,t)-flow)

effective resistance of d = optimum value/energy

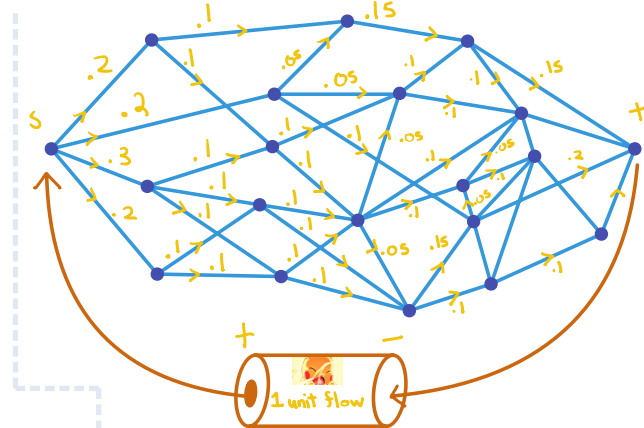
Ohm's law: $f = R^{-1}B^T p$
 for electric flow f , electric potentials p

$$L = BR^{-1}B^T \in \mathbb{R}^{V \times V}$$

random walk terminology:

Hitting time $H(s, t)$ = expected # steps until a random walk from s reaches t

Cover time $C(s)$ = expected # steps until a random walk from s visits every vertex



$$\text{electrical energy} = \sum_{e \in E} r_e f_e^2$$

electrical flow = least energy

def $B: \mathbb{R}^E \rightarrow \mathbb{R}^V$ by
 $(Bf)_v = \text{net flow at } v$
 $d_v = \begin{cases} +1 & \text{if } v=s \\ -1 & \text{if } v=t \\ 0 & \text{otherwise} \end{cases}$

$$\min_{f \in \mathbb{R}^E} \frac{1}{2} \sum_e r_e f_e^2 \text{ s.t. } Bf = d$$

(electrical energy) (f is (s,t)-flow)

effective resistance of d = optimum value/energy

Ohm's law: $f = R^{-1} B^T p$
 for electric flow f , electric potentials p

$$L = B R^{-1} B^T \in \mathbb{R}^{V \times V}$$

$$(\text{effective resist. of } d) = \langle d, L^{-1} d \rangle$$

$$H(t, t) = 0$$

$$H(u, t) = 1 + \frac{1}{\deg(u)} \sum_{x \in N(u)} H(x, t)$$

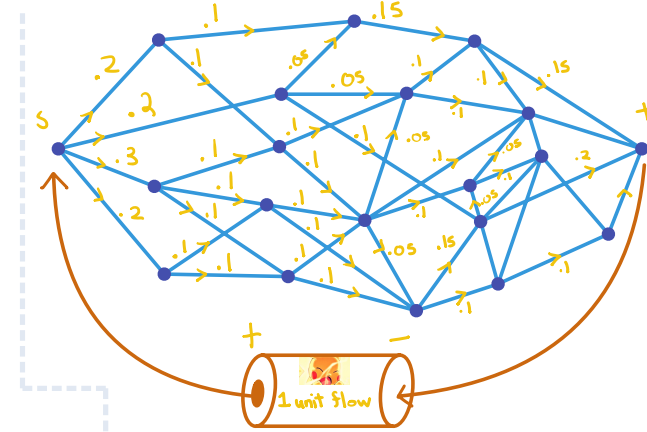
$$-p(u) = 1 - \frac{1}{\deg(u)} \sum_{v \in N(u)} p_v$$

$$\Leftrightarrow \sum_{(u,v) \in E} \underbrace{p_v - p_u}_{f(u,v)} = \deg(u)$$

$$\{f(u,v) = p_v - p_u\} \text{ routes}$$

$\deg(u)$ flow out of each $u \neq t$

$2m - \deg(t)$ flow into t



$$\text{electrical energy} = \sum_{e \in E} r_e f_e^2$$

electrical flow = least energy

def $B: \mathbb{R}^E \rightarrow \mathbb{R}^V$ by
 $(Bf)_v = \text{net flow at } v$

$$d_v = \begin{cases} +1 & \text{if } v=s \\ -1 & \text{if } v=t \\ 0 & \text{otherwise} \end{cases}$$

$$\min_{f \in \mathbb{R}^E} \frac{1}{2} \sum_e r_e f_e^2 \text{ s.t. } Bf = d$$

(electrical energy) (f is (s,t)-flow)

effective resistance of d = optimum value/energy

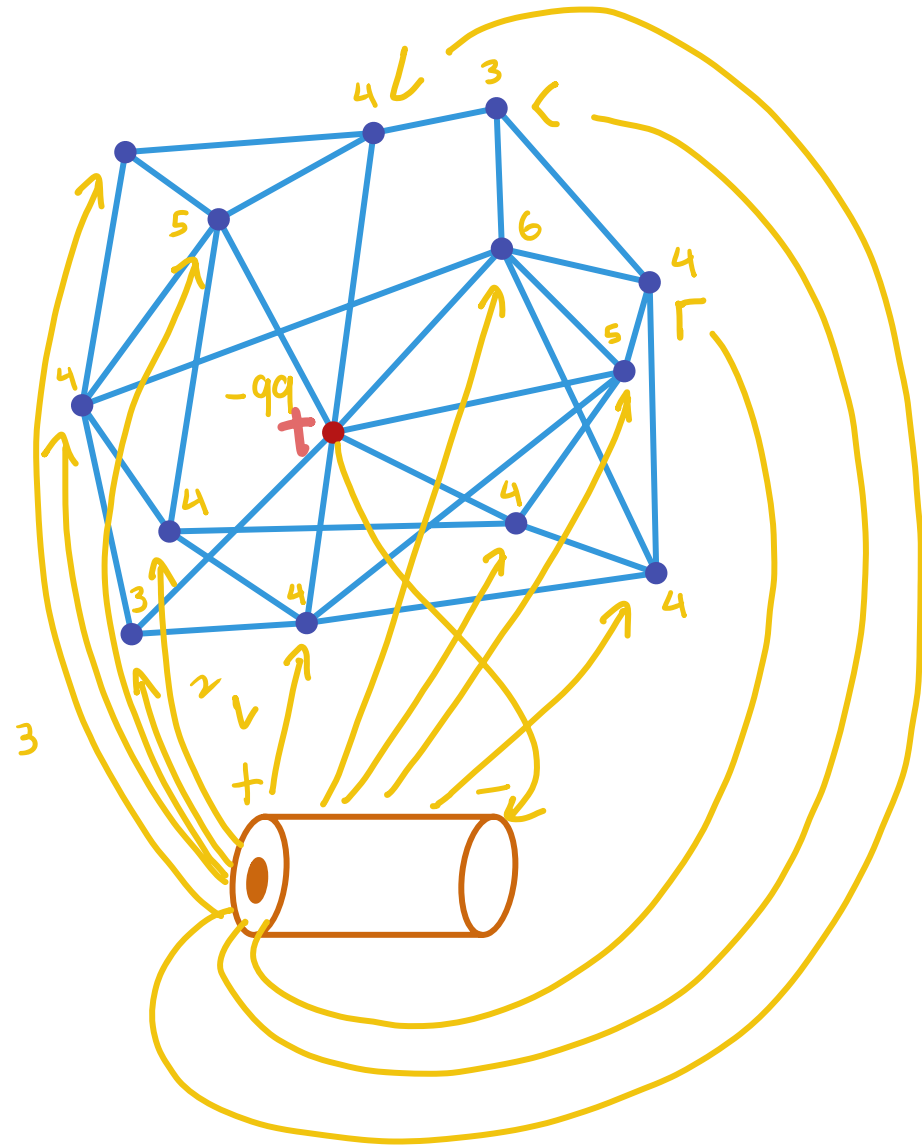
Ohm's law: $f = R^{-1} B^T p$
 for electric flow f , electric potentials p

$$L = B R^{-1} B^T \in \mathbb{R}^{V \times V}$$

$$(\text{effective resist. of } d) = \langle d, L^{-1} d \rangle$$

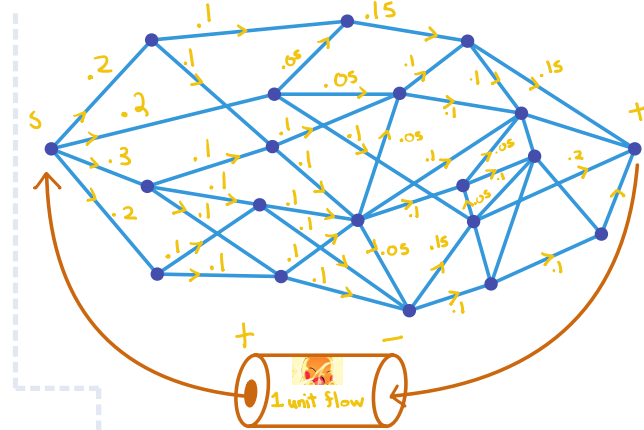
$\Rightarrow p$ encodes flow routing

$$d_v = \begin{cases} \deg(u) & v \neq t \\ (2m - \deg(t)) & v = t \end{cases}$$



Lemma

$$H(u,v) + H(v,u) = 2m \left(\begin{array}{l} \text{energy of unit} \\ (u,v)\text{-electric flow} \end{array} \right)$$



$$\text{electrical energy} = \sum_{e \in E} r_e f_e^2$$

electrical flow = least energy

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emma

$$H(u,v) + H(v,u) = 2m \begin{pmatrix} \text{energy of unit} \\ (u,v)\text{-electric flow} \end{pmatrix}$$

proof: let $p = -H(\cdot, u)$

$$q = -H(\cdot, v)$$

$d =$ demands for unit (u,v) -flow

$$r = p - q$$

$$p_u = 0 \quad q_v = 0$$

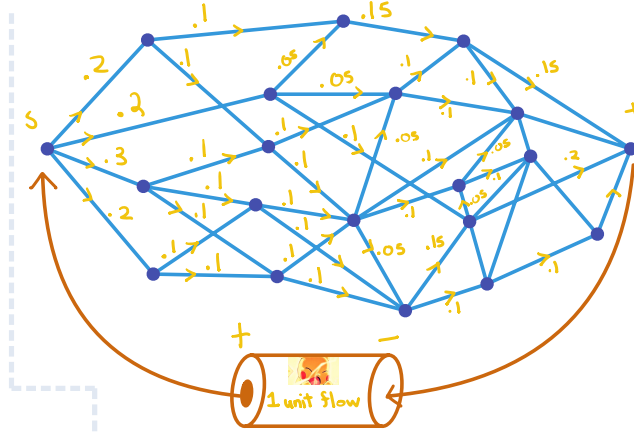
$$H(u,v) + H(v,u) = -p_v - q_u$$

$$= p_u - p_v + q_v - q_u = r_u - r_v = \langle r, d \rangle$$

Let $g(u,v) = p_v - p_u$, $h(u,v) = q_v - q_u$

$$f(u,v) = r_v - r_u = g - h$$

g routes: $\deg(x)$ out $\Rightarrow$ all $x \neq u$
all else into u



$$\text{electrical energy} = \sum_{e \in E} r_e f_e^2$$

electrical flow = flow of least energy

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$$\min_{f \in \mathbb{R}^E} \quad \frac{1}{2} \sum_e r_e f_e^2 \quad \text{s.t.} \quad Bf = d$$

(electrical energy) (f is (s)ti-slow)

effective resistance of d = optimum value/energy

Ohm's law: $\xi = R^{-1} B^T p$
for electric flow ξ , electric potentials p

$$L = BR^{-1}B^T \in \mathbb{R}^{V \times V}$$

$$(\text{effective resist. of } d) = \langle d, L^{-1}d \rangle$$

h routes: $\deg(x)$ out of all $x \neq v$
all else into v

f routes: $2m$ v to u

$$H(u,v) + H(v,u) = \langle r, d \rangle \stackrel{2md=Bf}{=} BB^T r$$

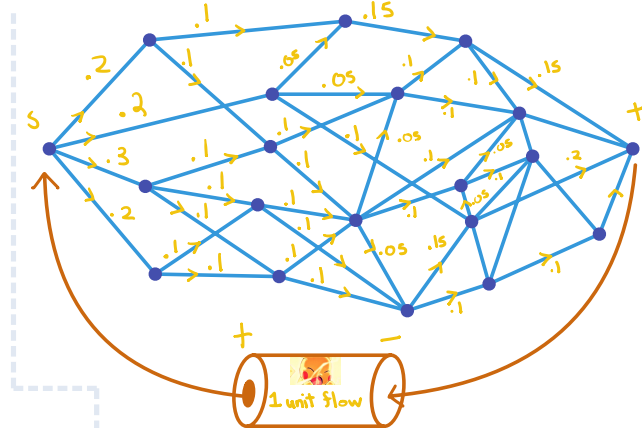
$$= \frac{1}{2m} \langle r, BB^T r \rangle$$

$$= \frac{1}{2m} \langle B^T r, B^T r \rangle$$

$$= \frac{1}{2m} \langle f, f \rangle$$

$$= 2m \left\langle \frac{f}{2m}, \frac{f}{2m} \right\rangle$$

$$= 2m \left(\begin{array}{l} \text{energy of unit} \\ (u,v)\text{-electric flow} \end{array} \right)$$



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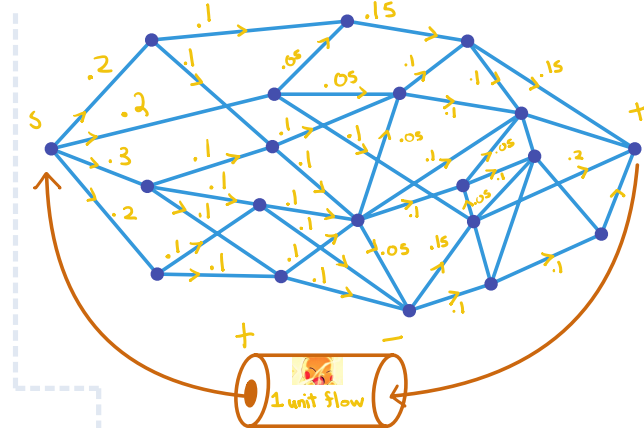
(electrical energy) (f is s(t)-flow)

effective resistance of d = optimum value/energy

Ohm's law: $f = R^{-1} B^T p$
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$$\left(\begin{array}{l} \text{effective} \\ \text{resist. of } d \end{array} \right) = \langle d, L^{-1}d \rangle$$



random walk terminology:

Hitting time $H(s,t)$ = expected # steps until a random walk from s reaches t

Cover time $C(s)$ = expected # steps until a random walk from s visits every vertex

$$\text{electrical energy} = \sum_{e \in E} r_e \xi_e^2$$

electrical flow = flow of least energy

def $B: \mathbb{R}^E \rightarrow \mathbb{R}^V$ by
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$$d_v = \begin{cases} +1 & \text{if } v = s \\ -1 & \text{if } v = t \\ 0 & \text{otherwise} \end{cases}$$

$$\min_{\mathbf{f} \in \mathbb{R}^E} \quad \frac{1}{2} \sum_e r_e f_e^2 \quad \text{s.t.} \quad \mathbf{B}\mathbf{f} = \mathbf{d}$$

(electrical energy) (f is (s)tr-Flow)

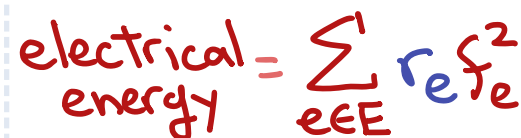
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Theorem $C(s) \leq 2mn$



electrical flow = flow of least energy

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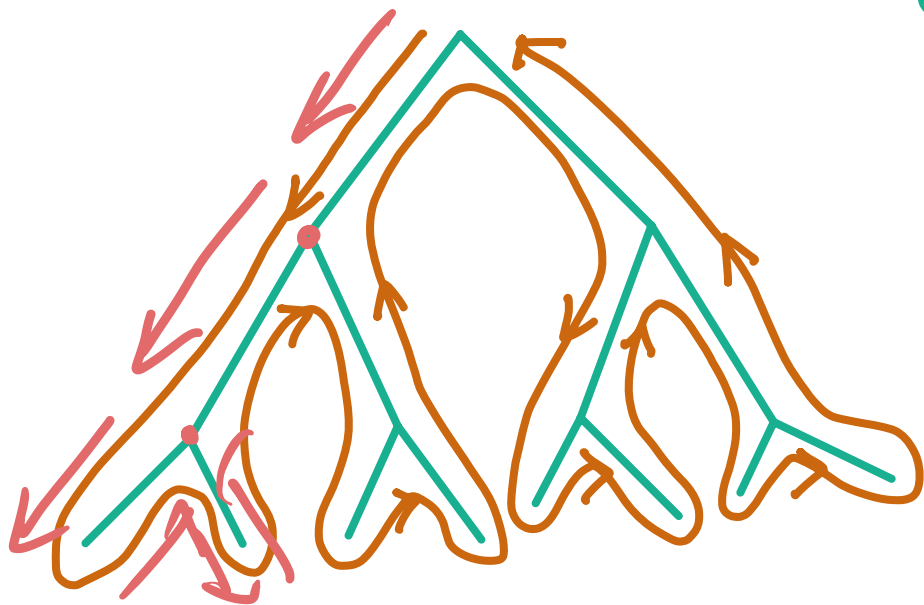
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Lemma $H(u,v) + H(v,u) = 2m$ (energy of unit (u,v) -electric flow)

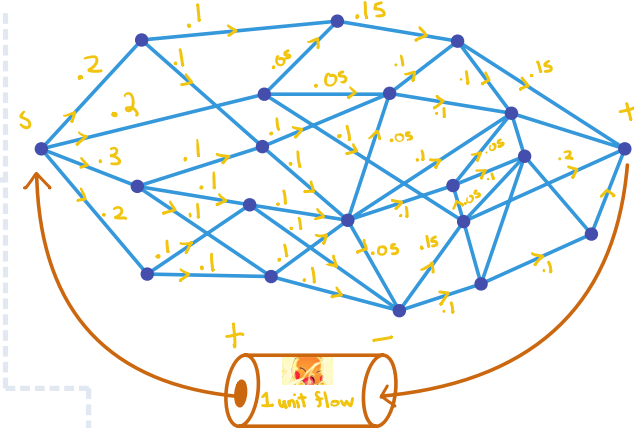
Theorem $C(s) \leq 2mn$

proof Six traversal of spanning tree



$$\text{total time} = \sum_{\{u,v\} \in T} H(u,v) + H(v,u)$$

$$= 2m \sum_{\{u,v\} \in T} (\text{energy of unit } (u,v)\text{-electric flow})$$



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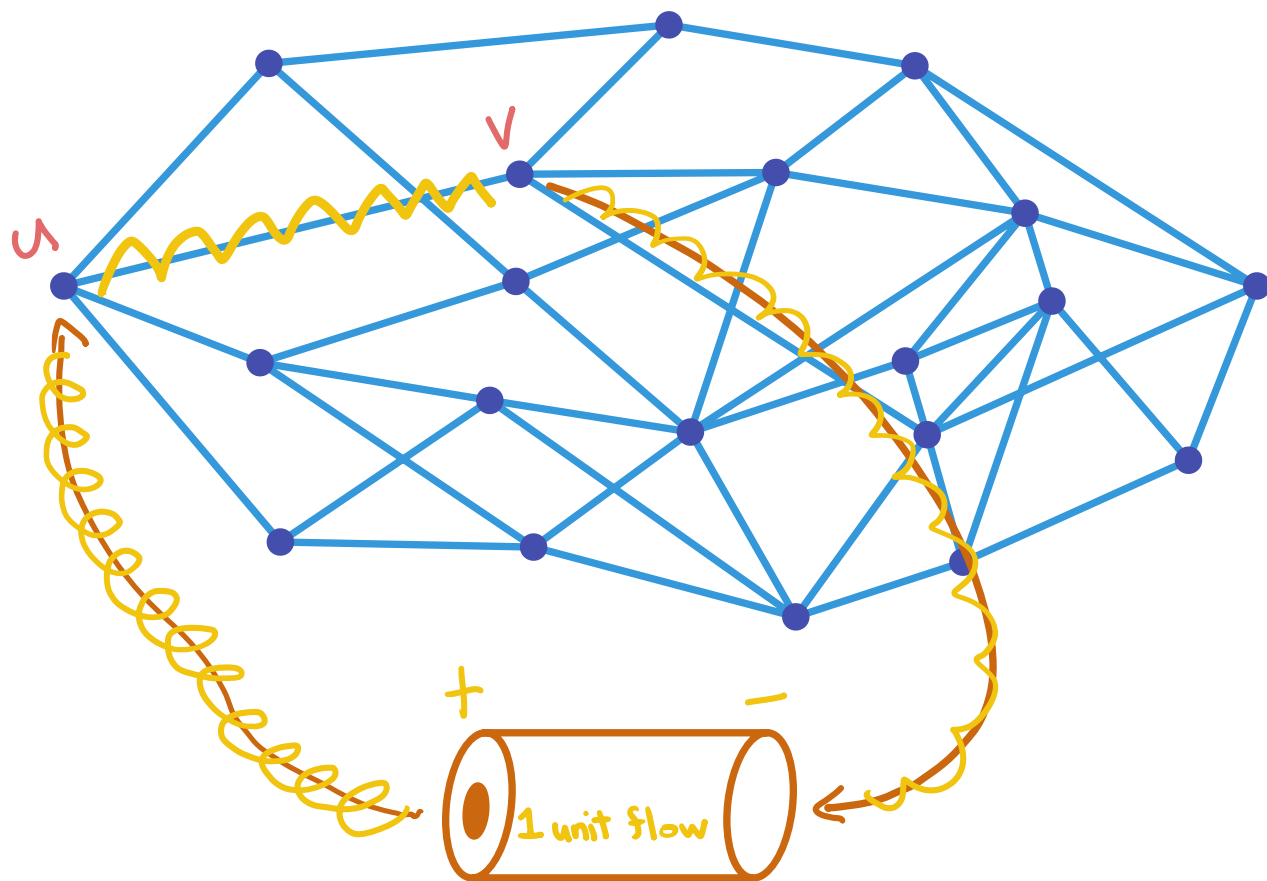
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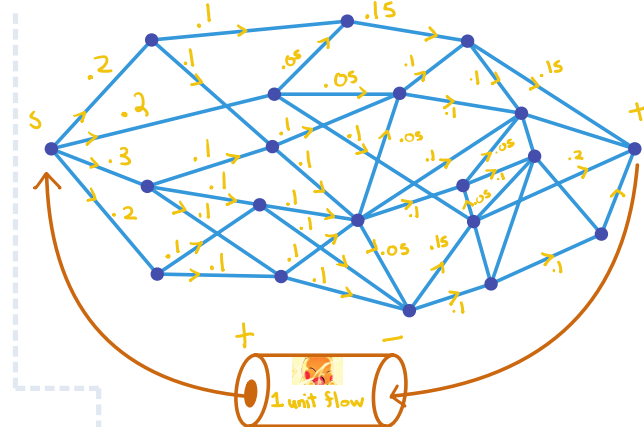
$$L = B R^{-1} B^T \in \mathbb{R}^{V \times V}$$

$$(\text{effective resist. of } d) = \langle d, L^{-1} d \rangle$$

$$\text{total time} = 2m \sum_{\{u,v\} \in E} (\text{energy of unit } (u,v)\text{-electric flow})$$



$$(\text{energy of unit } (u,v)\text{-electric flow}) \leq 1$$



$$\text{electrical energy} = \sum_{e \in E} r_e f_e^2$$

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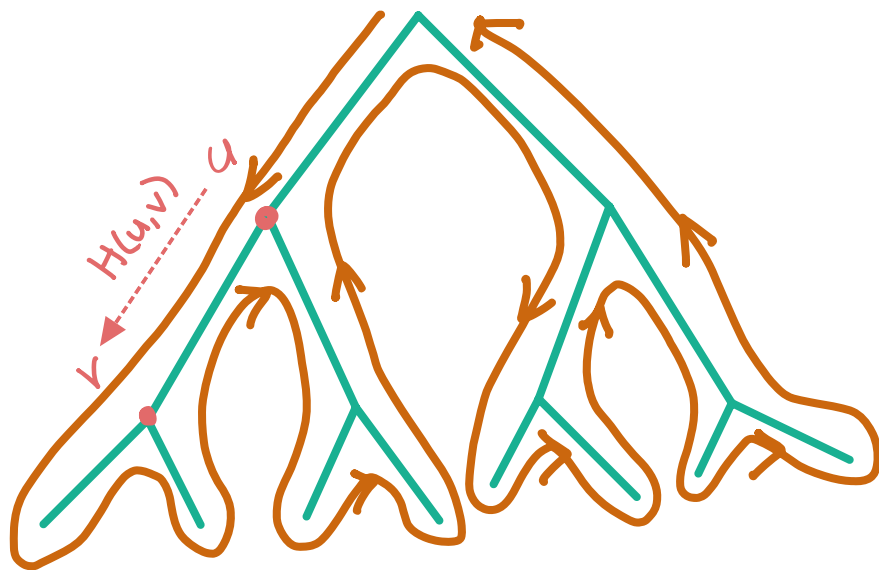
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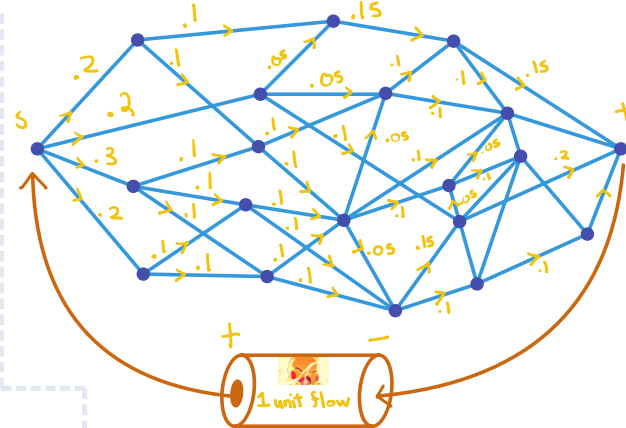
Theorem $C(s) \leq 2mn$

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$$\text{total time} = \sum_{\{u,v\} \in T} H(u,v) + H(v,u)$$

$$= 2m \sum_{\{u,v\} \in T} (\text{energy of unit } (u,v)\text{-electric flow}) \leq 2m(n-1)$$



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