

Connectivity & electricity

Lecture 16

Probability (and techniques)

Algorithms (models and applications)

Random Walks

Stationary Distribution

Perron-Frobenius Theorem

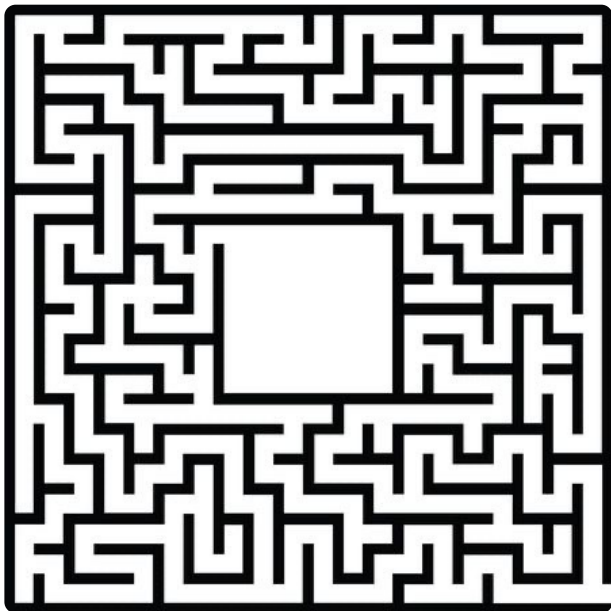
Pagerank

connectivity

Input: $G=(V,E)$ undirected,
(m edges, n vertices)

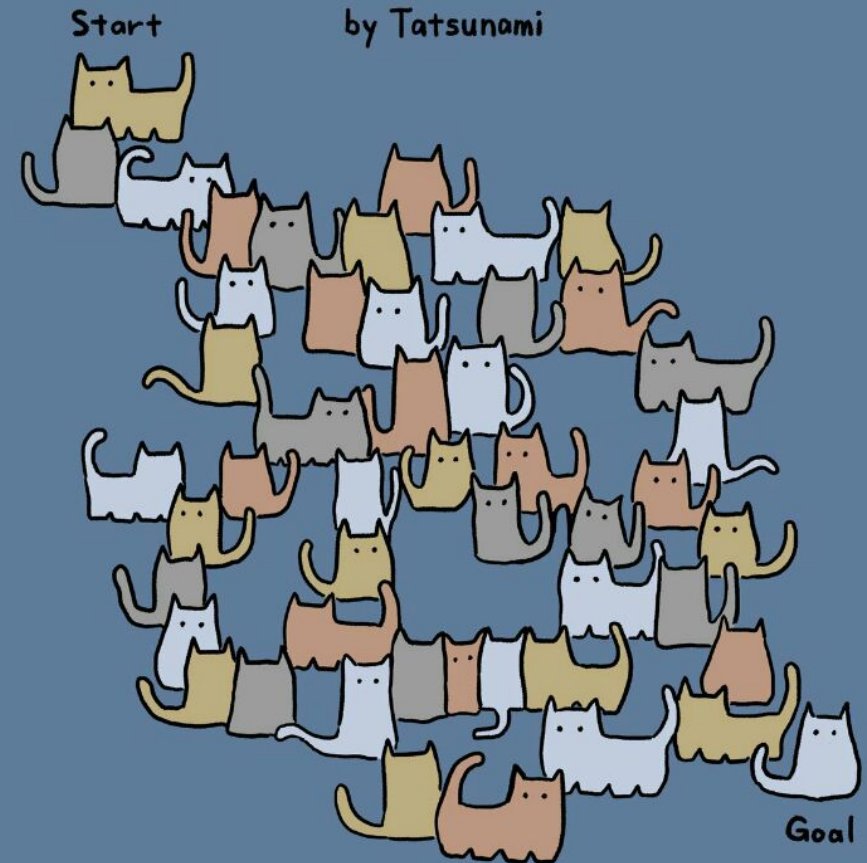
$s, t \in V$

Q: is there a path from
s to t?



CAT MAZE

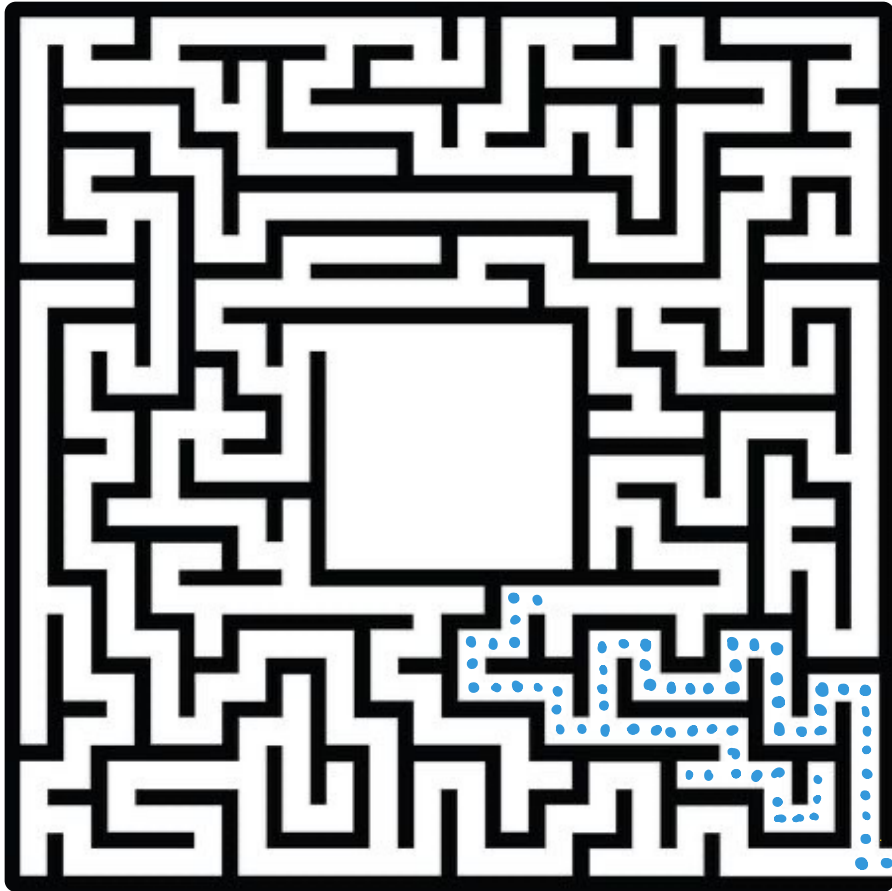
by Tatsunami



Reach the Goal by following the cats, alternating between cats facing the front and cats facing backwards.

The cats in your path must be connected or overlapping.

It doesn't matter what color or size they are.



Breadcrumbs

BFS, DFS take
linear time and space

can connectivity be solved in...

Yes

No

Not
voting

linear space?

BFS,
DFS

$O(\log^{O(1)} n)$ space?

Savitch

$O(\log n)$ space?

today

$O(\log n)$ space
deterministically?

in a
few
lectures!

Lecture 18

Probability
(and techniques)

Hitting times

Cover times

Electrical networks

Algorithms
(models and applications)

Connectivity
in $O(\log n)$ space

can connectivity be solved in $O(\log n)$ space?

algorithm:

(1) repeat $O(\log n)$ times:

(a) start at $s \in V$

(b) randomly walk for $O(mn)$ steps.

(c) if walk visits t , return TRUE

(2) return False

random walk terminology:

Hitting time $H(s,t)$ = expected # steps until a random walk from s reaches t

Cover time $C(s)$ = expected # steps until a random walk from s visits every vertex

can connectivity be solved in $O(\log n)$ space?

algorithm:

- (1) repeat $O(\log n)$ times:
 - (a) start at $s \in V$
 - (b) randomly walk for $O(mn)$ steps.
 - (c) if walk visits t , return TRUE
- (2) return False

random walk terminology:

Hitting time $H(s, t)$ = expected # steps until a random walk from s reaches t

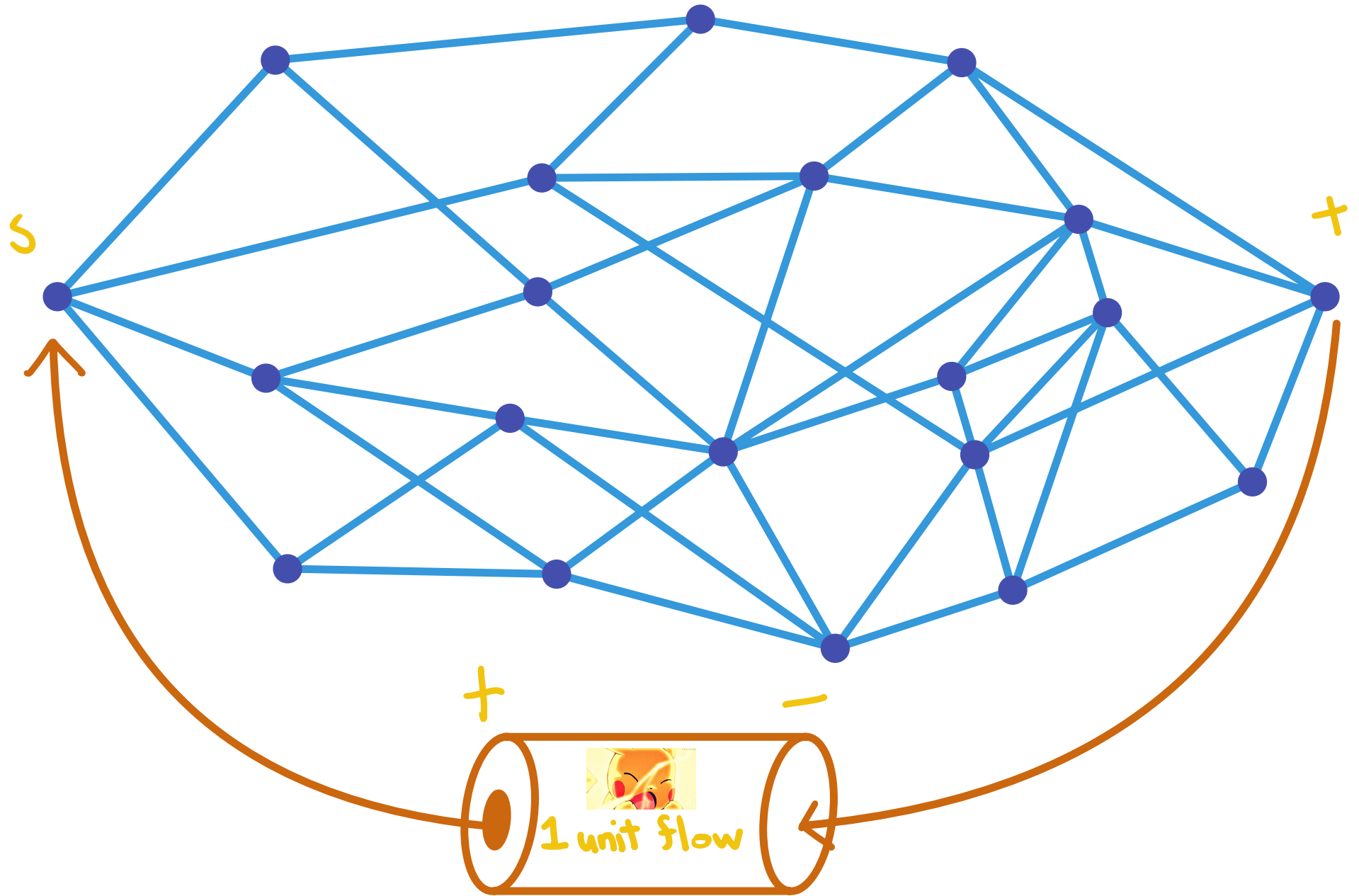
Cover time $C(s)$ = expected # steps until a random walk from s visits every vertex

Theorem

If G is connected, then $C(s) \leq m(n-1)$



Electrical networks



electricity takes the "path of least resistance"

"flow of least resistance"

each edge e has resistance $r_e > 0$

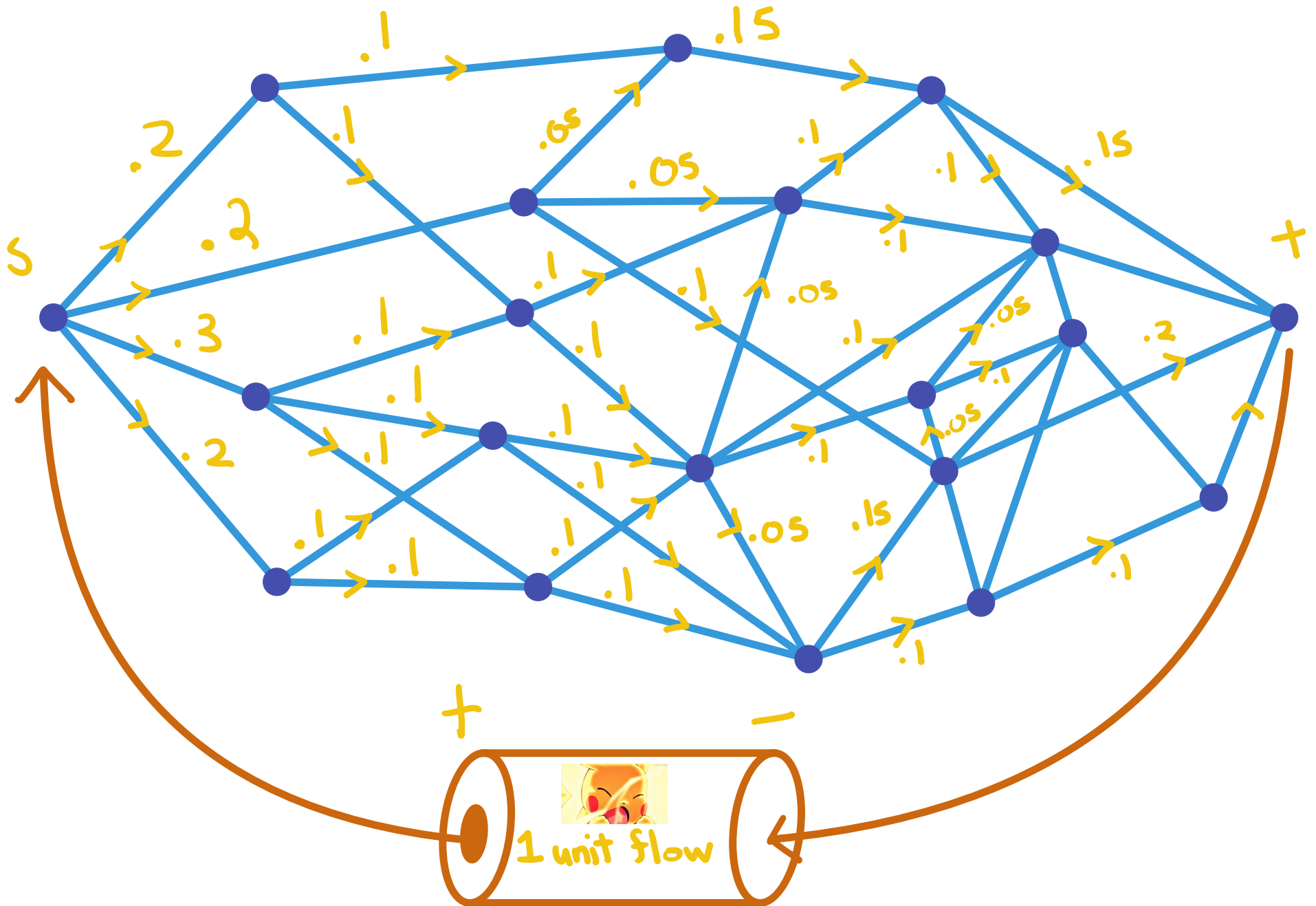
for flow $f: E \rightarrow \mathbb{R}_{\geq 0}$,

$$\text{electrical energy} = \sum_{e \in E} r_e f_e^2$$

electrical flow = flow of least energy

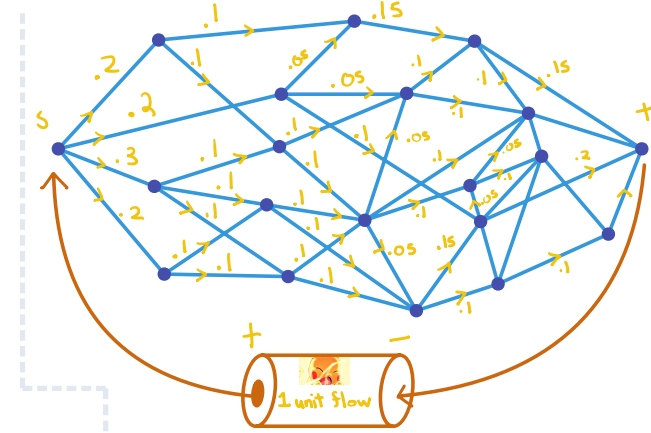
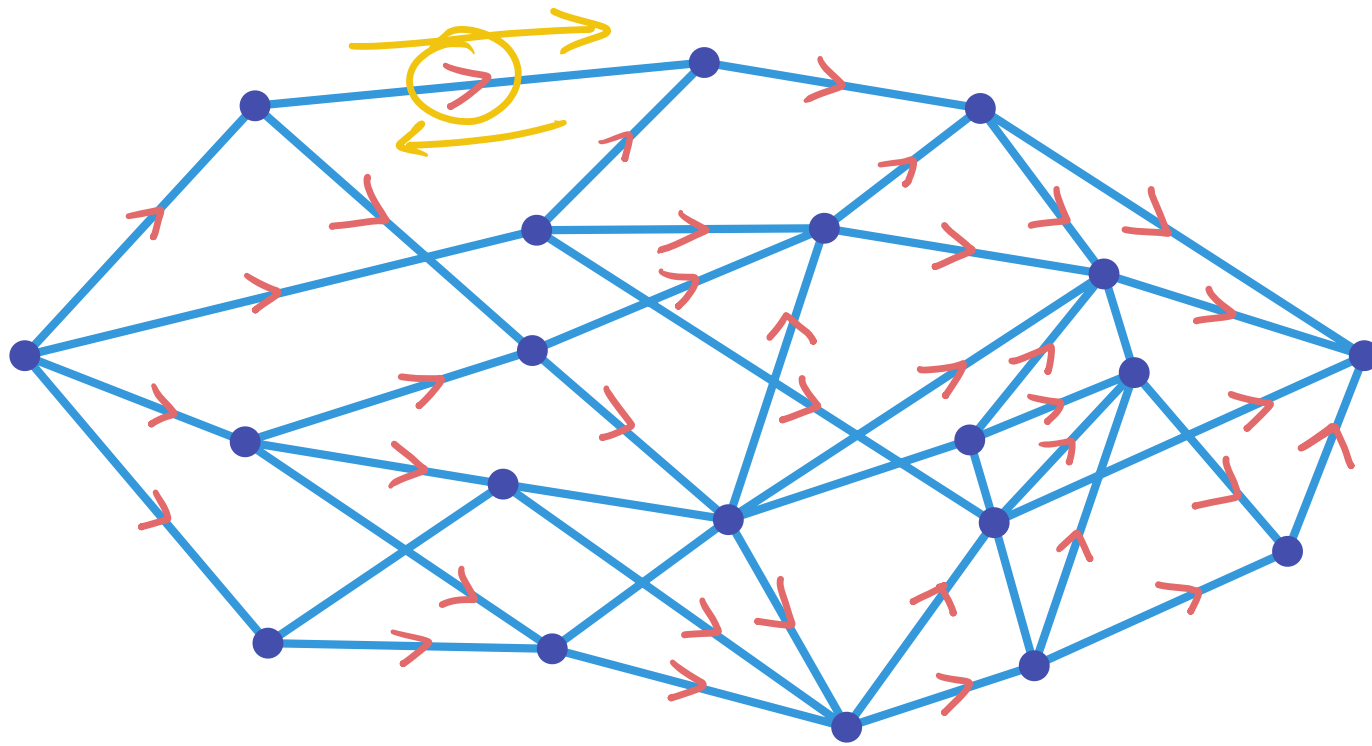
$$\text{electrical energy} = \sum_{e \in E} r_e f_e^2$$

electrical flow = flow of least energy



Electrical flow = convex minimization
math setup

orient edges arbitrarily:



$$\text{electrical energy} = \sum_{e \in E} r_e f_e^2$$

electrical flow = flow of least energy

$f_e > 0 \Rightarrow |f_e|$ units flow in direction of e

$f_e < 0 \Rightarrow |f_e|$ units flow in opposite direction

Electrical flow = convex minimization
math setup

Flow constraints = linear system

def $B: \mathbb{R}^E \rightarrow \mathbb{R}^V$ by
(flows)

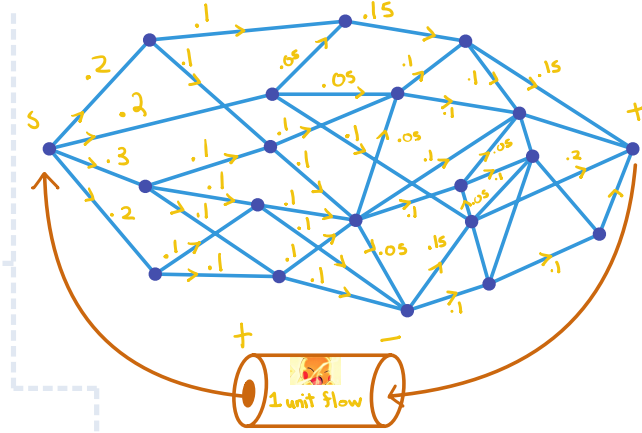
$(Bf)_v = \text{net flow at } v$

B is a linear function (check!)

$\Rightarrow B$ is a matrix $(\mathbb{R}^{V \times E})$

we want

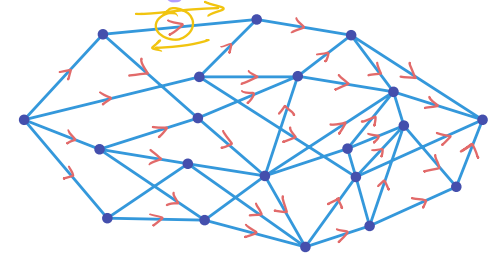
$$Bf = d \quad \text{where } d_v = \begin{cases} +1 & \text{if } v = s \\ -1 & \text{if } v = t \\ 0 & \text{otherwise} \end{cases}$$



$$\text{electrical energy} = \sum_{e \in E} r_e f_e^2$$

electrical flow = least energy

orient edges arbitrarily:



$f_e > 0 \Rightarrow |f_e|$ units flow in direction of e

$f_e < 0 \Rightarrow |f_e|$ units flow in opposite direction

Electrical flow = convex minimization

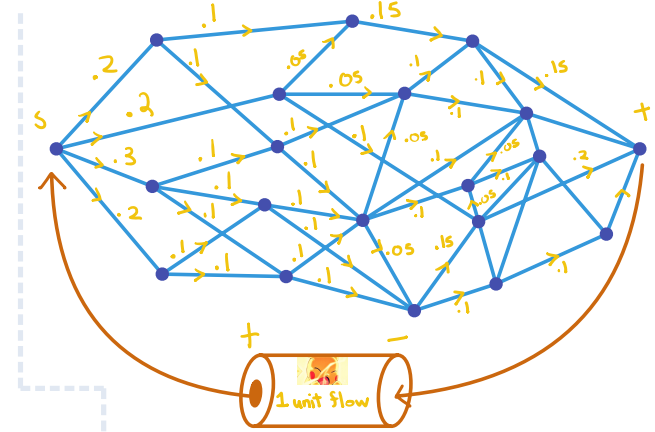
$$\min_{f \in \mathbb{R}^E} \quad \frac{1}{2} \sum_e r_e f_e^2$$

(electrical energy)

s.t. $Bf = d$
(f routes d)

electrical flow = optimum solution

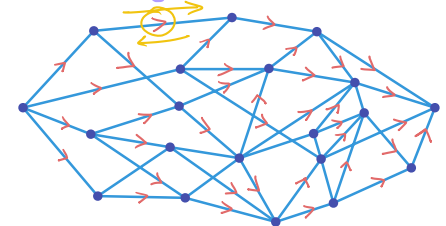
$\Rightarrow$ we should study (general)
optimality conditions



$$\text{electrical energy} = \sum_{e \in E} r_e f_e^2$$

electrical flow = flow of least energy

orient edges arbitrarily:



$f_e > 0 \Rightarrow |f_e|$ units slow in direction of e

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des $B: \mathbb{R}^E \rightarrow \mathbb{R}^V$ by

$$(BF)_v = \text{net flow at } v$$

B is a linear function (check!)

$\Rightarrow B$ is a matrix $(\mathbb{R}^{V \times E})$

we want $Bf = d$

where $d_v = \begin{cases} +1 & \text{if } v=s \\ -1 & \text{if } v=t \\ 0 & \text{otherwise} \end{cases}$

$$\min_{f \in \mathbb{R}^E} \frac{1}{2} \sum_e r_e f_e^2$$

(electrical energy)

$$\text{s.t. } Bf = d$$

(f routes d)

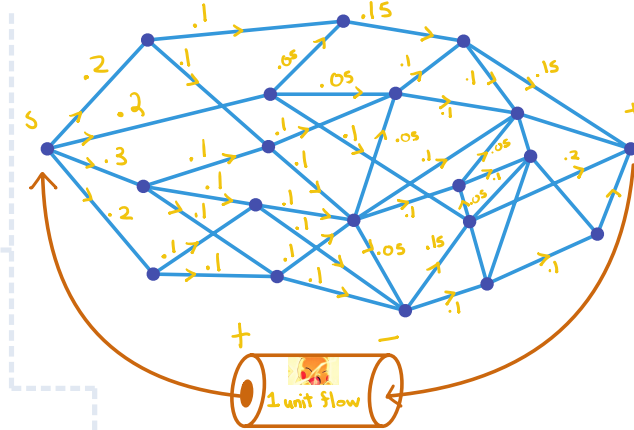
(First order conditions)

"Ohm's Law" let f route d .

f is the electrical flow

$$\Leftrightarrow f_e = \frac{p_v - p_u}{r_e} \quad \forall e = (u, v)$$

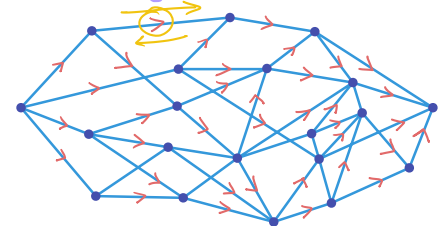
for some $p: V \rightarrow \mathbb{R}$
"vertex potentials"



$$\text{electrical energy} = \sum_{e \in E} r_e f_e^2$$

electrical flow = least energy

orient edges arbitrarily:



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def $B: \mathbb{R}^E \rightarrow \mathbb{R}^V$ by

$(Bf)_v = \text{net flow at } v$

B is a linear function (check!)

$\Rightarrow B$ is a matrix $(\mathbb{R}^{V \times E})$

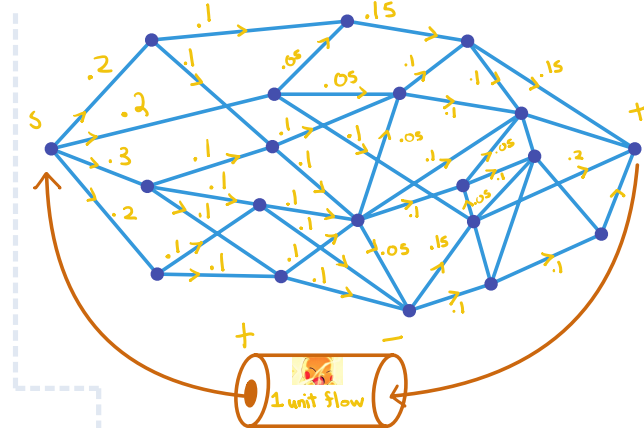
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(First order conditions)

"Ohm's Law" let f route d.

f is the electrical flow $\Leftrightarrow f_e = \frac{p_v - p_u}{r_e} \quad \forall e = (u, v)$
"vertex potentials"
for some $p: V \rightarrow \mathbb{R}$



Proof ($\Rightarrow$) let f be electrical flow

edge lengths $\ell(e) = r_e f_e$ (= gradient)

$$\text{electrical energy} = \sum_{e \in E} r_e f_e^2$$

electrical flow = least energy

claim: no negative cycles w/r/t ℓ

def.

potentials $p(v) = \min_s d_\ell(s, v)$ (shortest path distance w/r/t ℓ)

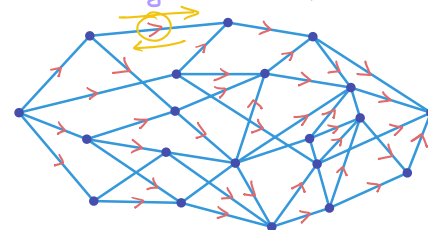
$$p(v) - p(u) \leq \ell(u, v) \quad (\text{triangle inequality})$$

$$p(u) - p(v) \leq \ell(v, u) \quad (\text{triangle inequality})$$

$$p_v - p_u = r_e f_e \quad e = (u, v)$$

$$\ell(u, v) = -\ell(v, u) = r_{uv} f(u, v) = -r_{uv} f(v, u)$$

orient edges arbitrarily:



$f_e > 0 \Rightarrow |f_e|$ units flow in direction of e

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def $B: \mathbb{R}^E \rightarrow \mathbb{R}^V$ by

$(Bf)_v = \text{net flow at } v$

B is a linear function (check!)

$\Rightarrow B$ is a matrix $(\mathbb{R}^{V \times E})$

we want $Bf = d$

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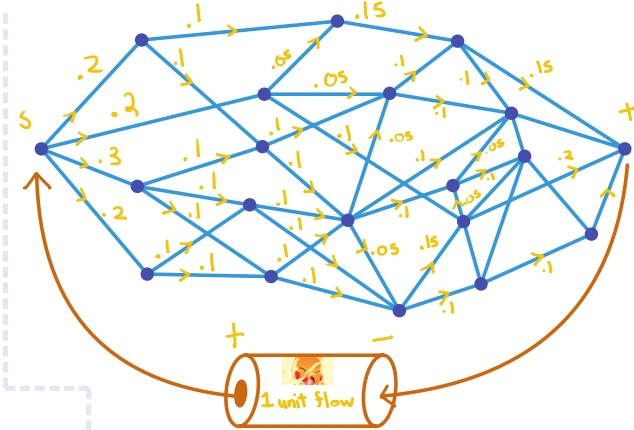
(First order conditions)

"Ohm's Law" let f route d .

f is the electrical flow

$$f_e = \frac{p_v - p_u}{r_e} \quad \forall e = (u, v)$$

"vertex potentials"
for some $p: V \rightarrow \mathbb{R}$



$$\text{electrical energy} = \sum_{e \in E} r_e f_e^2$$

electrical flow = least energy

Proof ($\Leftarrow$) suppose

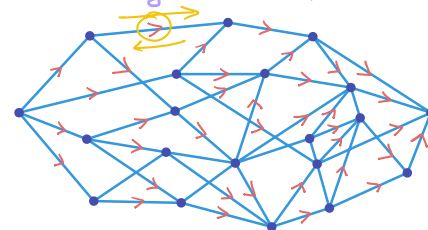
g = any flow routing d

$f - g$ is a circulation (net flow 0 everywhere)
(sum of cycles)

edge lengths $l(e) = r_e f_e = p_v - p_u \quad \forall e = (u, v)$
(gradient of f)

$$\text{cycle } C: l(C) = \sum_{uv \in C} l(u, v) = \sum_{uv \in C} p_v - p_u = 0$$

orient edges arbitrarily:



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def $B: \mathbb{R}^E \rightarrow \mathbb{R}^V$ by
(flows)

$(Bf)_v = \text{net flow at } v$

B is a linear function (check!)

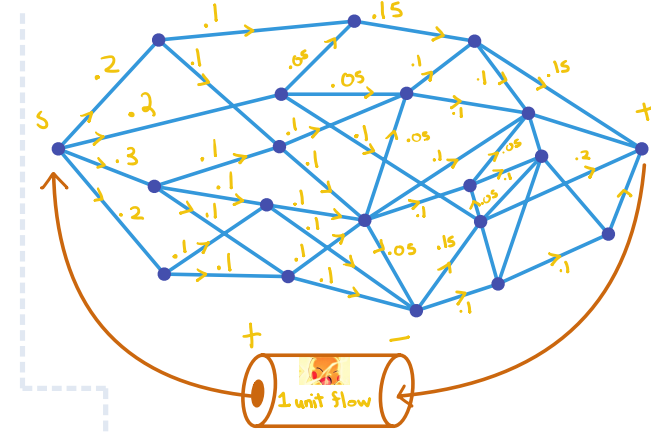
$\Rightarrow B$ is a matrix $(\mathbb{R}^{V \times E})$

we want $Bf = d$

where $d_v = \begin{cases} +1 & \text{if } v = s \\ -1 & \text{if } v = t \\ 0 & \text{otherwise} \end{cases}$

Convexity of $E(f) = \sum_e r_e f_e^2$:

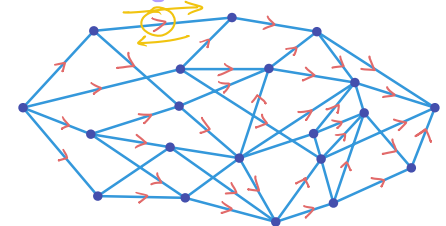
$$\begin{aligned}
 \mathcal{E}(g) &\geq \mathcal{E}(f) + \langle \mathcal{E}'(f), g-f \rangle \\
 &= \mathcal{E}(f) + \langle \overset{\text{circulation}}{L}, g-f \rangle \\
 &= \mathcal{E}(f)
 \end{aligned}$$



$$\text{electrical energy} = \sum_{e \in E} r_e f_e^2$$

electrical flow = least energy

orient edges arbitrarily:



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Electrical flow = convex minimization

$$\min_{f \in \mathbb{R}^E} \quad \frac{1}{2} \sum_e r_e f_e^2$$

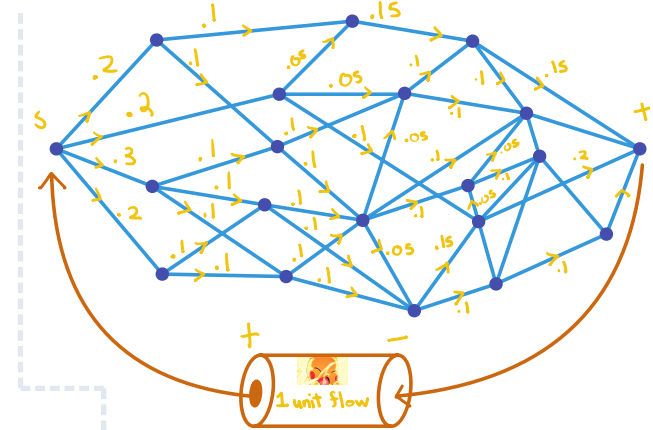
(electrical energy)

s.t. $Bf = d$

$(f \text{ is } (s,t)\text{-flow})$

electrical flow = optimum solution

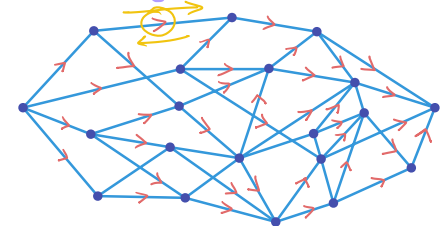
"effective" resistance of d = optimum value/energy



$$\text{electrical energy} = \sum_{e \in E} r_e f_e^2$$

electrical flow = flow of least energy

orient edges arbitrarily:



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def B: $\mathbb{R}^E \rightarrow \mathbb{R}^V$ by

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B is a linear function (check!)

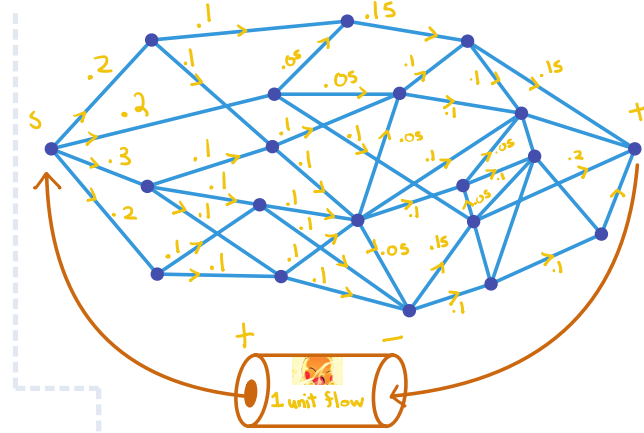
$\Rightarrow B$ is a matrix $(\mathbb{R}^{V \times E})$

we want $Bf = d$

where $d_v = \begin{cases} +1 & \text{if } v=s \\ -1 & \text{if } v=t \\ 0 & \text{otherwise} \end{cases}$

Lemma: let f route d (i.e., $Bf=d$)

f is electrical flow $\Leftrightarrow f = R^{-1}B^T p$ for some $p \in \mathbb{R}^V$
(vertex potentials)



$$\text{electrical energy} = \sum_{e \in E} r_e f_e^2$$

electrical flow = least energy

def $B: \mathbb{R}^E \rightarrow \mathbb{R}^V$ by
(Bf)_v = net flow at v
 $d_v = \begin{cases} +1 & \text{if } v=s \\ -1 & \text{if } v=t \\ 0 & \text{otherwise} \end{cases}$

$$\min_{f \in \mathbb{R}^E} \frac{1}{2} \sum_e r_e f_e^2 \text{ s.t. } Bf=d$$

(electrical energy) (f is (s,t)-flow)

effective resistance of d = optimum value/energy

Lemma consider problem

$$\min f(x) \text{ s.t. } Ax=b$$

w/ f convex, smoothly differentiable.

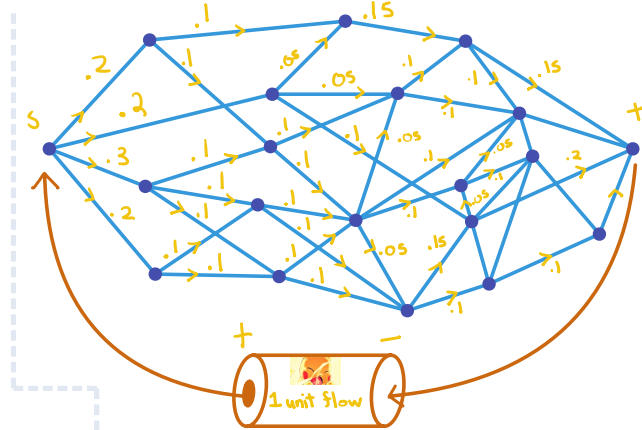
let x^* be opt. solution. then

$$f'(x^*) = A^T y \text{ for some } y$$

Lemma: let f route d (i.e., $Bf=d$)

f is
electrical flow $\Leftrightarrow f = R^{-1}B^T p$
for some $p \in \mathbb{R}^V$
(vertex potentials)

suppose f is electrical flow



$$\text{electrical energy} = \sum_{e \in E} r_e f_e^2$$

electrical flow = flow of least energy

def $B: \mathbb{R}^E \rightarrow \mathbb{R}^V$ by
(Bf) _{v} = net flow at v
 $d_v = \begin{cases} +1 & \text{if } v=s \\ -1 & \text{if } v=t \\ 0 & \text{otherwise} \end{cases}$

$$\min_{f \in \mathbb{R}^E} \frac{1}{2} \sum_e r_e f_e^2 \text{ s.t. } Bf=d$$

(electrical energy) (f is (s,t)-flow)

effective resistance of d = optimum value/energy

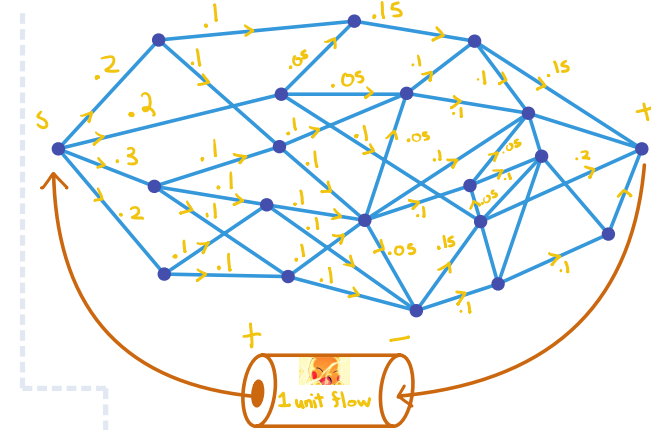
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$$\text{electrical energy} = \sum_{e \in E} r_e x_e^2$$

def $B: \mathbb{R}^E \rightarrow \mathbb{R}^V$ by

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effective resistance of d = optimum value/energy

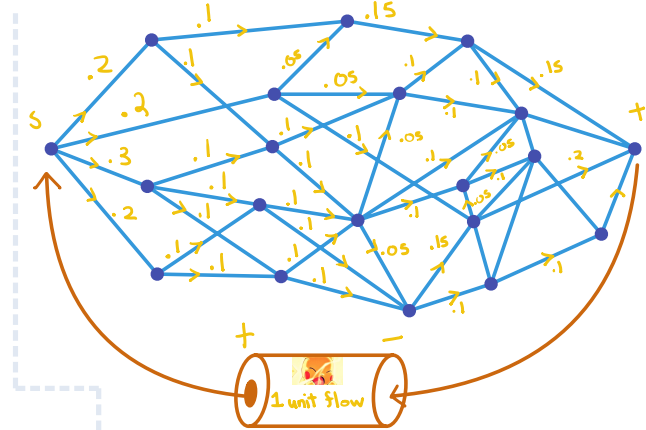
$$\min f(x) \quad \text{s.t.} \quad Ax=b$$

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Lemma: let f route d (i.e., $Bf=d$)

f is electrical flow $\Leftrightarrow f = R^{-1} B^T p$ for some $p \in \mathbb{R}^V$
(vertex potentials)

Suppose $f = R^{-1} B^T p$ for some $p \in \mathbb{R}^V$



$$\text{electrical energy} = \sum_{e \in E} r_e f_e^2$$

electrical flow = flow of least energy

def B: $\mathbb{R}^E \rightarrow \mathbb{R}^V$ by
 $(Bf)_v = \text{net flow at } v$

$$d_v = \begin{cases} +1 & \text{if } v = s \\ -1 & \text{if } v = t \\ 0 & \text{otherwise} \end{cases}$$

$$\min_{f \in \mathbb{R}^E} \quad \frac{1}{2} \sum_e r_e f_e^2 \quad \text{s.t.} \quad Bf = d$$

(electrical energy) (f is (s,t)-flow)

effective resistance of d = optimum value/energy

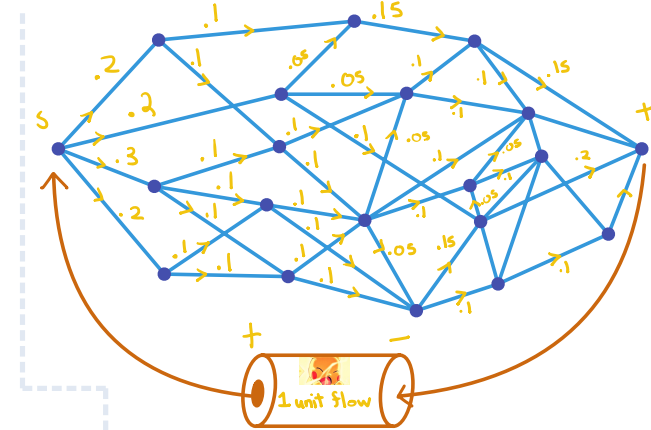
Lemma consider problem

$$\min f(x) \quad \text{s.t.} \quad Ax=b$$

w/ f convex, smoothly differentiable.

let x^* be opt. solution. then

$$f'(x^*) = A^T \gamma \text{ for some } \gamma$$



$$\text{electrical energy} = \sum_{e \in E} r_e f_e^2$$

electrical flow = least energy

def $B: \mathbb{R}^E \rightarrow \mathbb{R}^V$ by
 $(Bf)_v = \text{net flow at } v$
 $d_v = \begin{cases} +1 & \text{if } v=s \\ -1 & \text{if } v=t \\ 0 & \text{otherwise} \end{cases}$

$$\min_{f \in \mathbb{R}^E} \frac{1}{2} \sum_e r_e f_e^2 \text{ s.t. } Bf = d$$

(electrical energy) (f is (s,t)-flow)

effective resistance of d = optimum value/energy

Lemma consider problem

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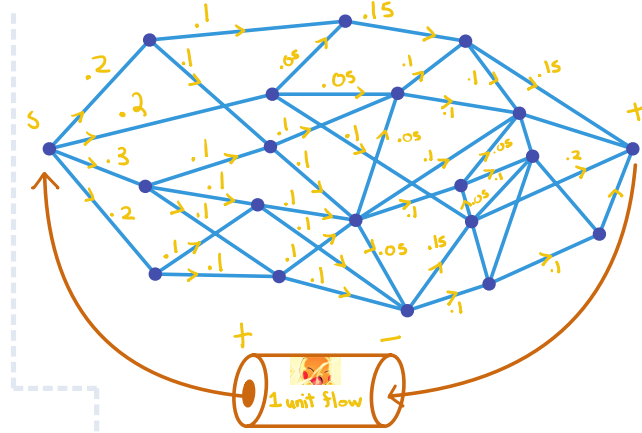
Lemma: let f route d (i.e., $Bf=d$)

f is electrical flow $\Leftrightarrow f = R^{-1}B^T p$ for some $p \in \mathbb{R}^V$
(vertex potentials)

$p \in \mathbb{R}^V$ called "electric potentials"

Ohm's law: $f = R^{-1}B^T p$

for electric flow f , electric potentials p



$$\text{electrical energy} = \sum_{e \in E} r_e f_e^2$$

electrical flow = flow of least energy

def $B: \mathbb{R}^E \rightarrow \mathbb{R}^V$ by
(Bf) _{v} = net flow at v
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$$\min_{f \in \mathbb{R}^E} \frac{1}{2} \sum_e r_e f_e^2 \text{ s.t. } Bf=d$$

(electrical energy) (f is (s,t)-flow)

effective resistance of d = optimum value/energy

Lemma consider problem

$$\min f(x) \text{ s.t. } Ax=b$$

w/ f convex, smoothly differentiable.

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Laplacian

$$L = BR^{-1}B^T \in \mathbb{R}^{V \times V}$$

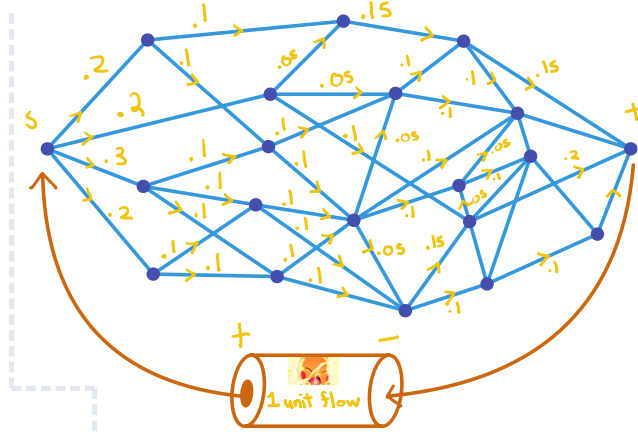
"Laplacian" of graph w/ edge weights $w(e) = \frac{1}{r(e)}$

e.g.

$$\langle x, Lx \rangle = \langle B^T x, R^{-1} B^T x \rangle$$

$$= \sum_{e \in E} \frac{1}{r_e} (B^T x)_e$$

$$= \sum_{e=\{u,v\}} \frac{1}{r_e} (x_u - x_v)^2$$



$$\text{electrical energy} = \sum_{e \in E} r_e f_e^2$$

electrical flow = flow of least energy

def $B: \mathbb{R}^E \rightarrow \mathbb{R}^V$ by

$$(BF)_v = \text{net flow at } v$$

$$d_v = \begin{cases} +1 & \text{if } v=s \\ -1 & \text{if } v=t \\ 0 & \text{otherwise} \end{cases}$$

$$\min_{\mathbf{f} \in \mathbb{R}^E} \quad \frac{1}{2} \sum_e r_e f_e^2 \quad \text{s.t.} \quad \mathbf{B}\mathbf{f} = \mathbf{d}$$

(electrical energy) (f is (s)ti-slow)

effective resistance of d = optimum value/energy

Ohm's law: $\xi = R^{-1} B^T p$
 for electric flow ξ , electric potentials p

Laplacian

$$L = BR^{-1}B^T \in \mathbb{R}^{V \times V}$$

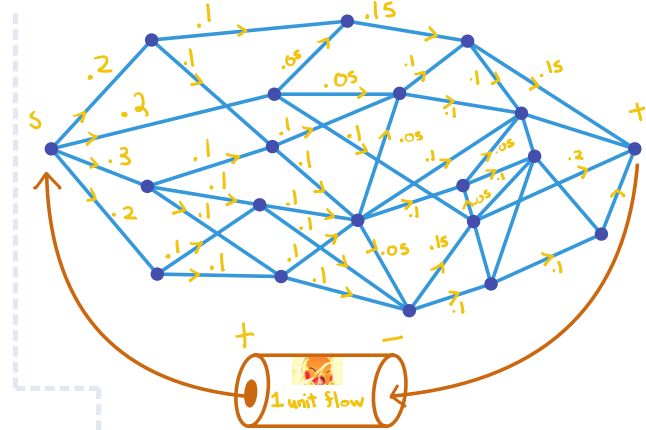
"Laplacian" of graph w/ edge weights $w(e) = \frac{1}{r(e)}$

$$\checkmark \langle x, Lx \rangle = \sum_{e=\{u,v\}} \frac{1}{r_e} (x_u - x_v)^2$$

for $S \subseteq V$, let $x = \frac{1}{2}(\mathbb{1}_S - \mathbb{1}_{\bar{S}})$

$$\langle x, Lx \rangle = \sum_{\substack{e \in \partial(S) \\ e = \{u, v\}}} \frac{1}{r_e} \left(\frac{1}{2} - \left(-\frac{1}{2} \right) \right)^2$$

$$= \sum_{e \in \partial(S)} \frac{1}{r_e} = w(\partial(S))$$



$$\text{electrical energy} = \sum_{e \in E} r_e f_e^2$$

electrical flow = flow of least energy

def $B: \mathbb{R}^E \rightarrow \mathbb{R}^V$ by
 $(Bf)_v = \text{net flow at } v$

$$d_v = \begin{cases} +1 & \text{if } v=s \\ -1 & \text{if } v=t \\ 0 & \text{otherwise} \end{cases}$$

$$\min_{\mathbf{f} \in \mathbb{R}^E} \quad \frac{1}{2} \sum_e r_e f_e^2 \quad \text{s.t.} \quad \mathbf{B}\mathbf{f} = \mathbf{d}$$

(electrical energy) (f is (s)ti-slow)

effective resistance of d = optimum value/energy

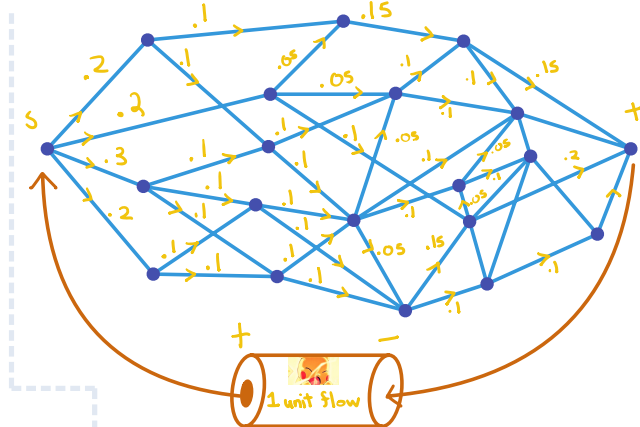
Ohm's law: $\xi = R^{-1} B^T p$
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Lemma $Lp=d$, and

$$(\text{effective resist. of } d) = \langle d, L^{-1}d \rangle = \langle p, d \rangle$$

Proof

electric potentials for d



$$\text{electrical energy} = \sum_{e \in E} r_e f_e^2$$

electrical flow = least energy

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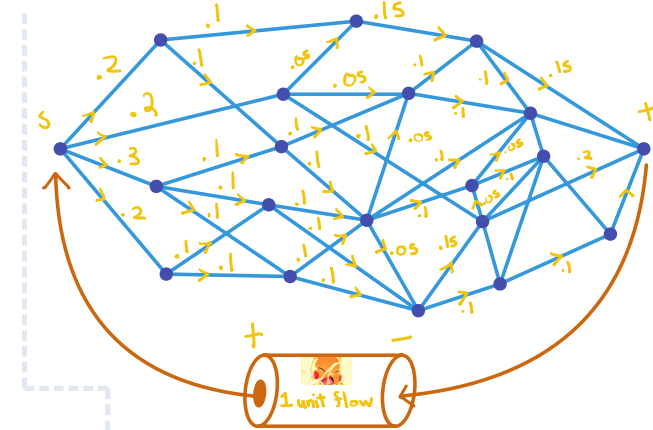
$$\min_{f \in \mathbb{R}^E} \frac{1}{2} \sum_e r_e f_e^2 \text{ s.t. } Bf = d$$

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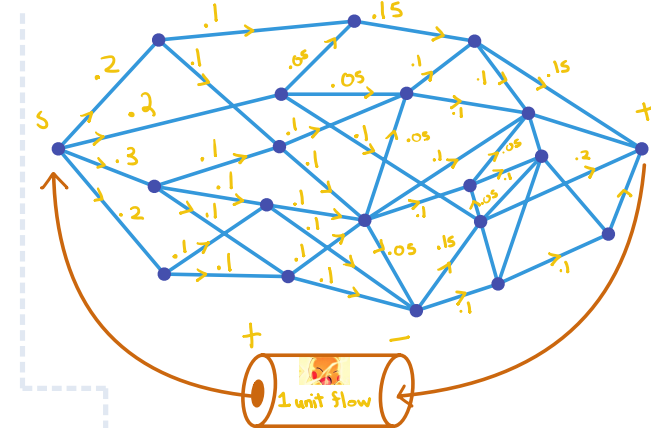
$$L = B R^{-1} B^T \in \mathbb{R}^{V \times V}$$

(effective resist. of d) = $\langle d, L^{-1} d \rangle$

random walk terminology:

Hitting time $H(s, t) =$ expected # steps until a random walk from s reaches t

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Hitting time $H(s,t)$ = expected # steps until a random walk from s reaches t

fix $t \in V$.

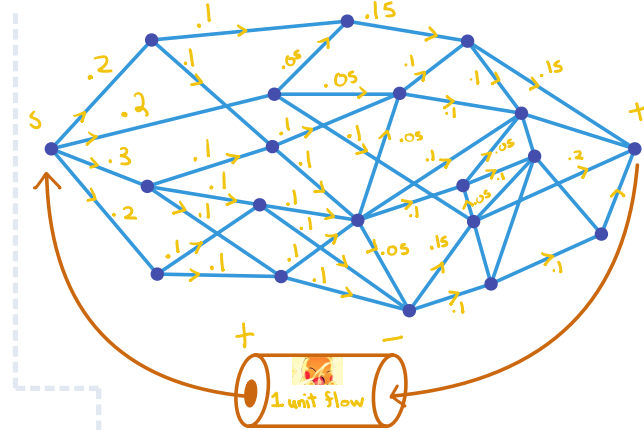
$$H(t,t) = 0$$

$$H(u,t) = 1 + \frac{1}{\deg(u)} \sum_{x \in N(u)} H(x,t)$$

define $p \in \mathbb{R}^V$ by $p(u) = -H(u,t) \quad \forall u$

$$\text{let } f = B^T p$$

$$(f(u,v) = p(v) - p(u) \quad \forall (u,v) \in \vec{E})$$



$$\text{electrical energy} = \sum_{e \in E} r_e f_e^2$$

electrical flow = least energy

def $B: \mathbb{R}^E \rightarrow \mathbb{R}^V$ by
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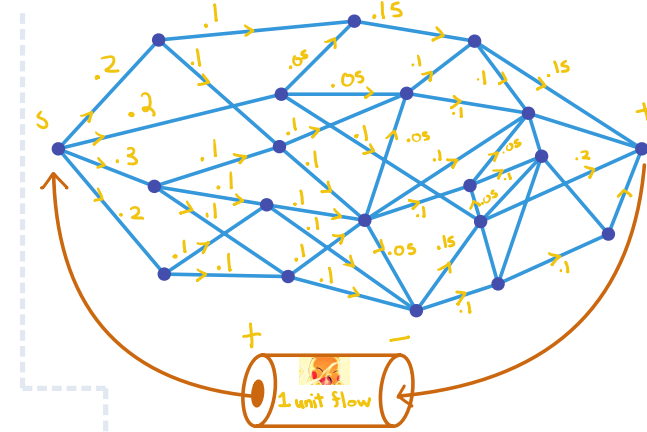
$$-p(u) = 1 - \frac{1}{\deg(u)} \sum_{v \in N(u)} p_v$$

$$\sum_{(u,v) \in E} \underbrace{p_v - p_u}_{f(u,v)} = \deg(u)$$

$$\{f(u,v) = p_v - p_u\} \text{ routes}$$

$\deg(u)$ flow out of each $u \neq t$

$2m - \deg(t)$ flow into t



$$\text{electrical energy} = \sum_{e \in E} r_e f_e^2$$

electrical flow = least energy

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(electrical energy) (f is (s,t)-flow)

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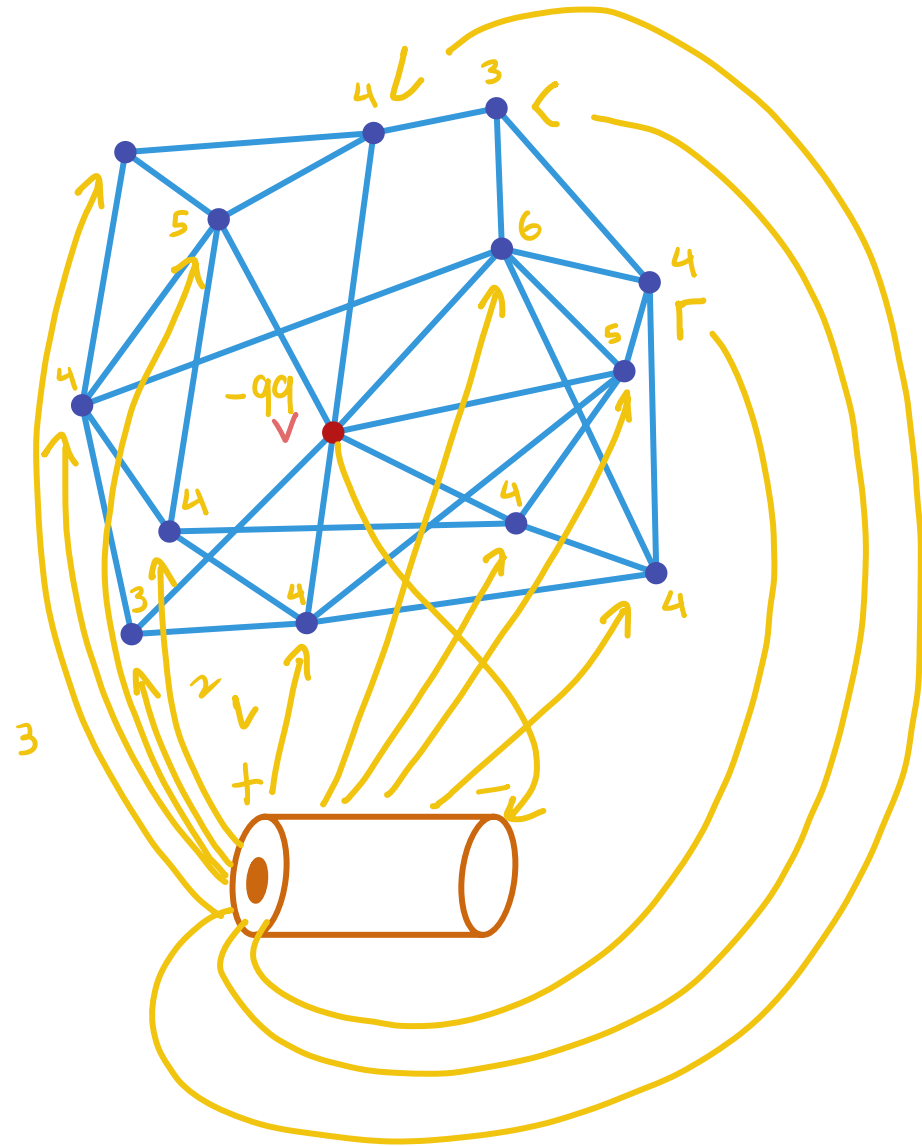
Ohm's law: $f = R^{-1} B^T p$
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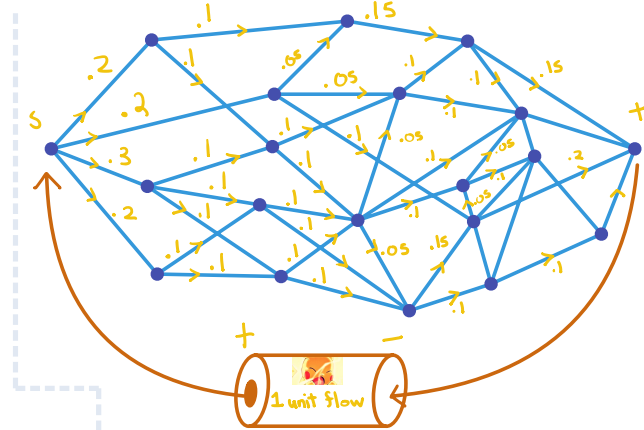
$\Rightarrow p$ encodes flow routing

$$d_v = \begin{cases} \deg(u) & v \neq t \\ (2m - \deg(t)) & v = t \end{cases}$$



Lemma

$$H(u,v) + H(v,u) = 2m \left(\begin{array}{l} \text{energy of unit} \\ (u,v)\text{-electric flow} \end{array} \right)$$



$$\text{electrical energy} = \sum_{e \in E} r_e f_e^2$$

electrical flow = least energy

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(electrical energy) (f is (s,t) -flow)

effective resistance of d = optimum value/energy

Ohm's law: $f = R^{-1} B^T p$
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$$L = B R^{-1} B^T \in \mathbb{R}^{V \times V}$$

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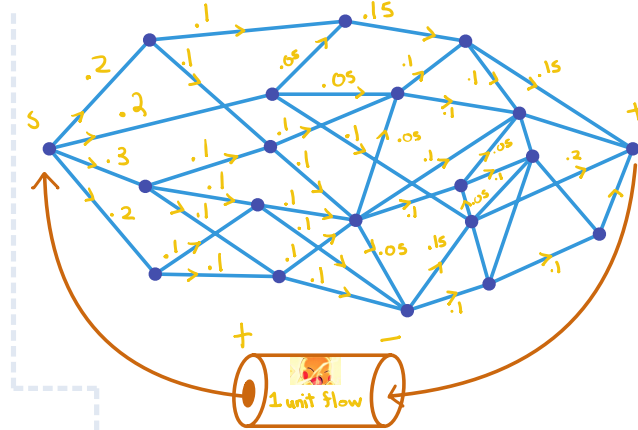
proof: let $p = -H(\cdot, u)$
 $q = -H(\cdot, v)$ $d = \begin{array}{l} \text{demands} \\ \text{for unit} \\ (u,v)\text{-flow} \end{array}$
 $r = p - q$

$$H(u,v) + H(v,u) = -p_v - q_u$$

$$= p_u - p_v + q_v - q_u = r_u - r_v = \langle r, d \rangle$$

Let $g(u,v) = p_v - p_u$, $h(u,v) = q_v - q_u$
 $f(u,v) = r_v - r_u = g - h$

g routes:



$$\text{electrical energy} = \sum_{e \in E} r_e f_e^2$$

electrical flow = least energy

def $B: \mathbb{R}^E \rightarrow \mathbb{R}^V$ by
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(electrical energy) (f is (s,t)-flow)

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h routes:

f routes:

$$H(u,v) + H(v,u) = \langle r, d \rangle$$

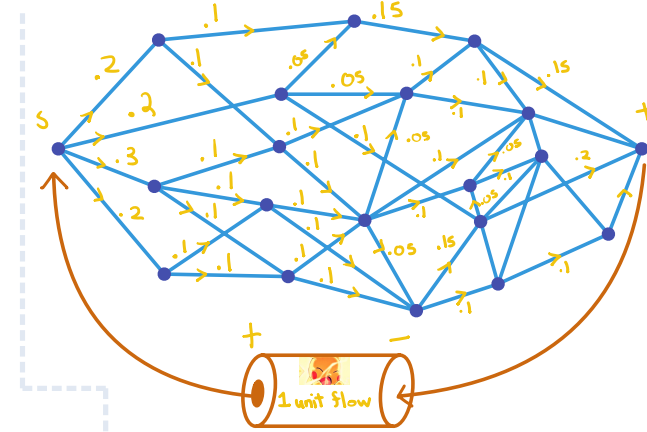
$$= \frac{1}{2m} \langle r, BB^T r \rangle$$

$$= \frac{1}{2m} \langle B^T r, B^T r \rangle$$

$$= \frac{1}{2m} \langle f, f \rangle$$

$$= 2m \left\langle \frac{f}{2m}, \frac{f}{2m} \right\rangle$$

$$= 2m \left(\begin{array}{l} \text{energy of unit} \\ (u,v)\text{-electric flow} \end{array} \right)$$



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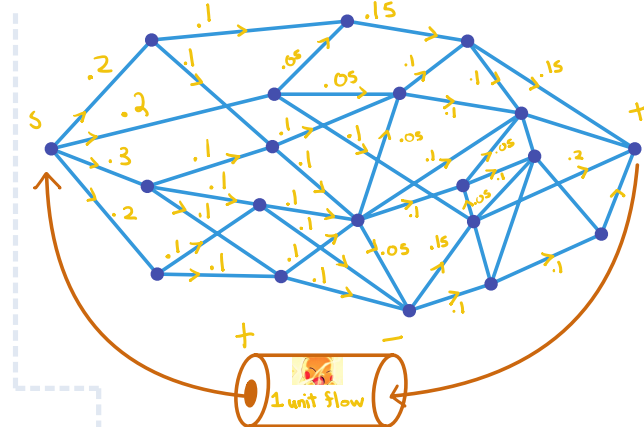
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(electrical energy) (f is (s,t)-flow)

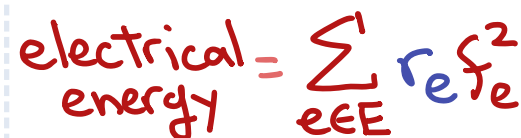
effective resistance of d = optimum value/energy

Ohm's law: $f = R^{-1} B^T p$
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$$L = B R^{-1} B^T \in \mathbb{R}^{V \times V}$$

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Theorem $C(s) \leq 2mn$



electrical flow = flow of least energy

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(electrical energy) (S is (st)-Slow)

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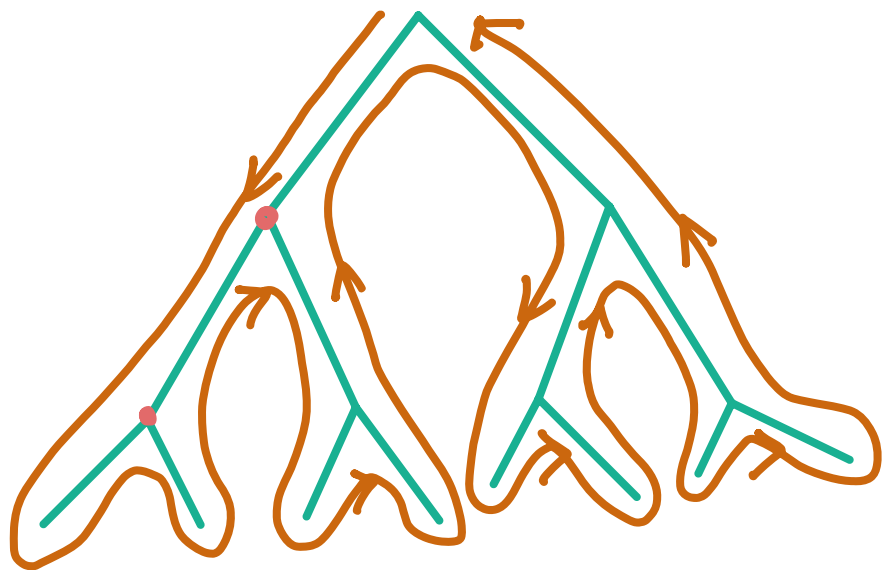
$$L = BR^{-1}B^T \in \mathbb{R}^{V \times V}$$

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Lemma $H(u,v) + H(v,u) = 2m$ (energy of unit (u,v) -electric flow)

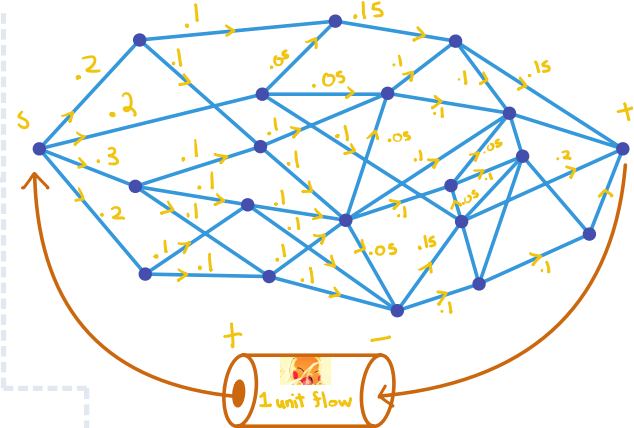
Theorem $C(s) \leq 2mn$

proof Six traversal of spanning tree



$$\text{total time} = \sum_{\{u,v\} \in T} H(u,v) + H(v,u)$$

$$= 2m \sum_{\{u,v\} \in T} (\text{energy of unit } (u,v)\text{-electric flow})$$



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(electrical energy) (f is (s,t)-flow)

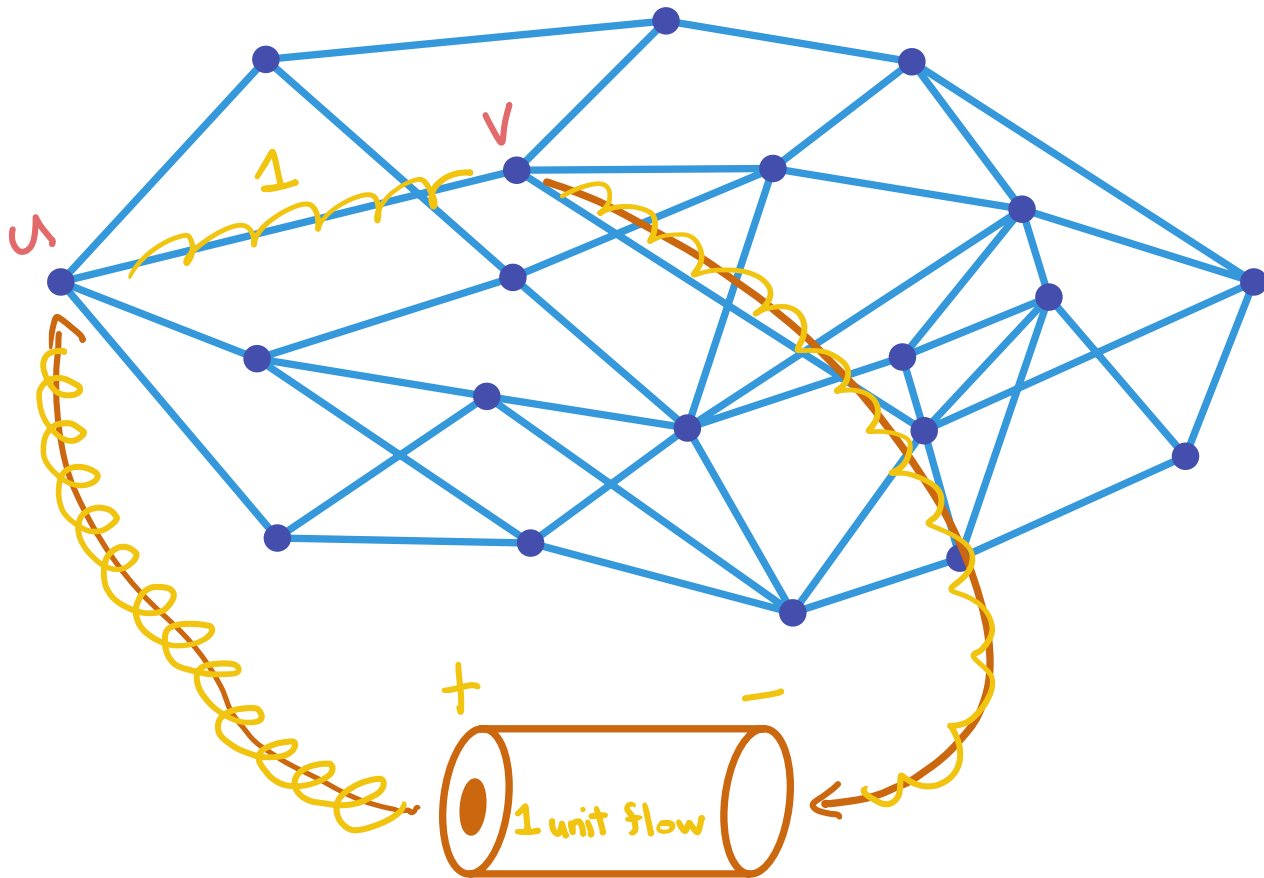
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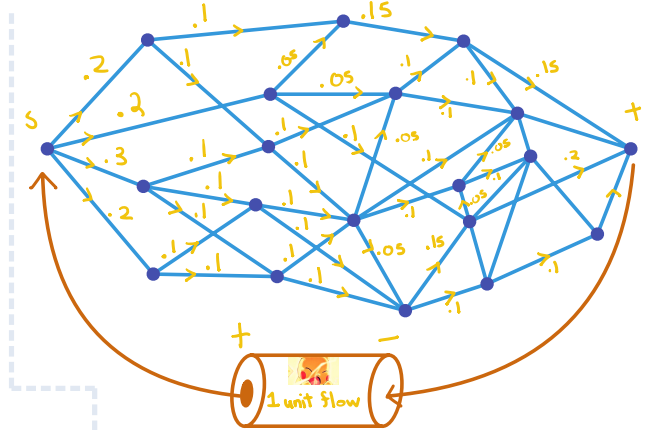
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$$\text{total time} = 2m \sum_{\{u,v\} \in E} (\text{energy of unit } (u,v)\text{-electric flow})$$



$$\frac{(\text{energy of unit})}{(u,v)\text{-electric flow}} \leq$$



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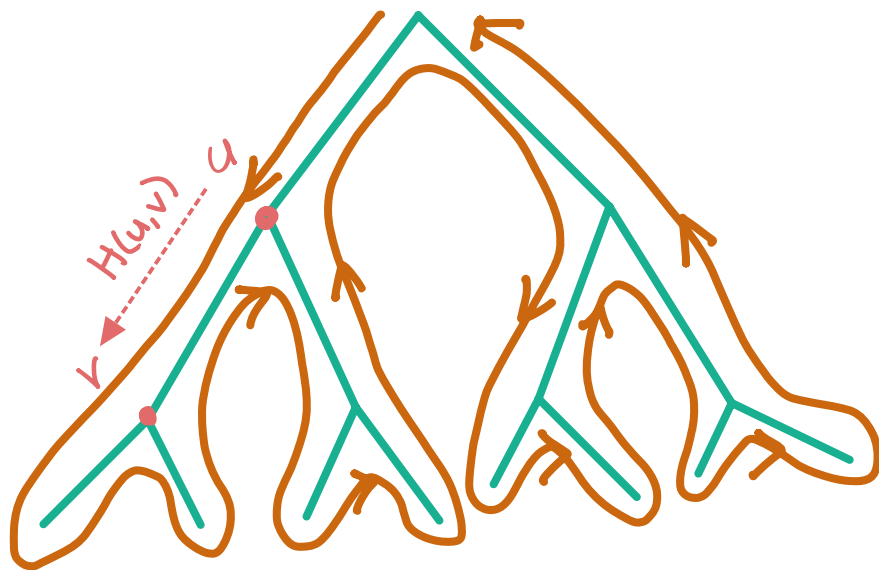
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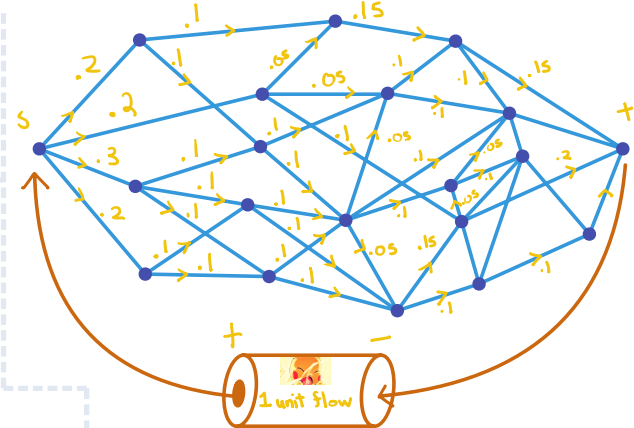
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