

Conductance + Mixing Time

Probability (and techniques)

Algorithms (models and applications)

Random Walks

Stationary Distribution

Perron-Frobenius Theorem

Pagerank

Probability (and techniques)

(and techniques)

Algorithms (models and applications)

(models and applications)

Hitting times

Cover times

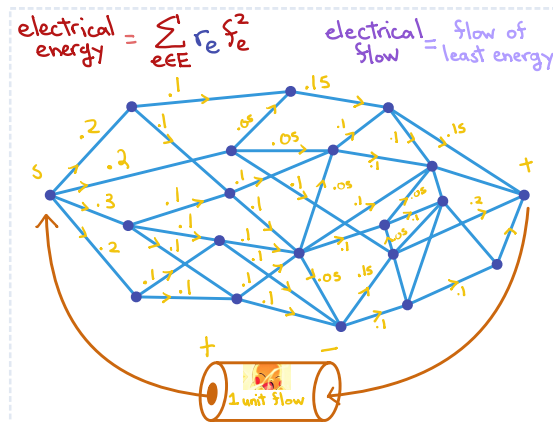
Electrical networks

Connectivity in $O(\log n)$ space

random walk terminology:

Hitting time $H(s,t)$ = expected # steps until a random walk from s reaches t

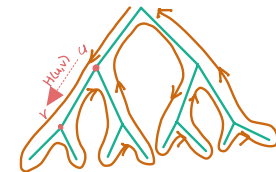
Cover time $C(s)$ = expected # steps until a random walk from s visits every vertex



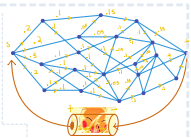
Lemma $H(u,v) + H(v,u) = 2m$ (energy of unit (u,v) -electric flow)

Theorem $C(s) \leq 2mn$

proof Six traversal of spanning tree



$$\begin{aligned} \text{total time} &= \sum_{\{u,v\} \in E} H(u,v) + H(v,u) \\ &= 2m \sum_{\{u,v\} \in E} (\text{energy of unit } (u,v)\text{-electric flow}) \end{aligned}$$



$$\text{electrical energy} = \sum_{e \in E} r_e f_e^2$$

electrical flow = flow of least energy

def B: $\mathbb{R}^E \rightarrow \mathbb{R}^V$ by
 $(Bf)_v = \text{net flow at } v$
 $d_v = \begin{cases} +1 & \text{if } v \text{ is } v_{\text{es}} \\ -1 & \text{if } v \text{ is } v_{\text{st}} \\ 0 & \text{otherwise} \end{cases}$

$$\min_{f \in \mathbb{R}^E} \frac{1}{2} \sum_e r_e f_e^2 \quad \text{s.t.} \quad Bf = d$$

effective resistance of d = optimum value/energy

Ohm's law: $\mathbf{S} = \mathbf{R}^{-1} \mathbf{B}^T \mathbf{p}$
 For electric flow $\mathbf{S}$, electric potentials $\mathbf{p}$

$$(\text{effective resist. of } d) = \langle d, L^{-1} d \rangle$$

Probability (and techniques)

Algorithms (models and applications)

Spectral theorem
for symmetric maps

Spectral analysis of
random walks

Let R have eigenvalues

$$\mu_1 = 1 \geq \mu_2 \geq \dots \geq \mu_n \geq -1$$

Theorem. let $\gamma = 1 - \max\{\mu_2, |\mu_n|\}$

$$\|R^k x - s\| \leq (1 - \gamma)^k \sqrt{\frac{\Delta_{\max}}{\Delta_{\min}}}$$

↑
stationary
distribution

Lecture 19

Probability (and techniques)

Algorithms (models and applications)

Spectral theorem
for symmetric maps

Spectral analysis of
random walks

Lecture 20

Probability
(and techniques)

Algorithms
(models and applications)

Conductance

Cheeger's inequality

Conductance-based
mixing time

Fiedler's algorithm
for low-conductance
cuts

Recall

Sparsest cut

Input: undirected graph $G=(V,E)$
edge capacities $c(e) > 0$ for $e \in E$
demands $b(u,v) \geq 0$ for $u,v \in V$ (symmetric)

Goal: compute SCV minimizing

"sparsity" $\frac{c(\delta(S))}{\sum_{u \in S, v \in \bar{S}} b(u,v)}$

← cut few edges

← split up many pairs

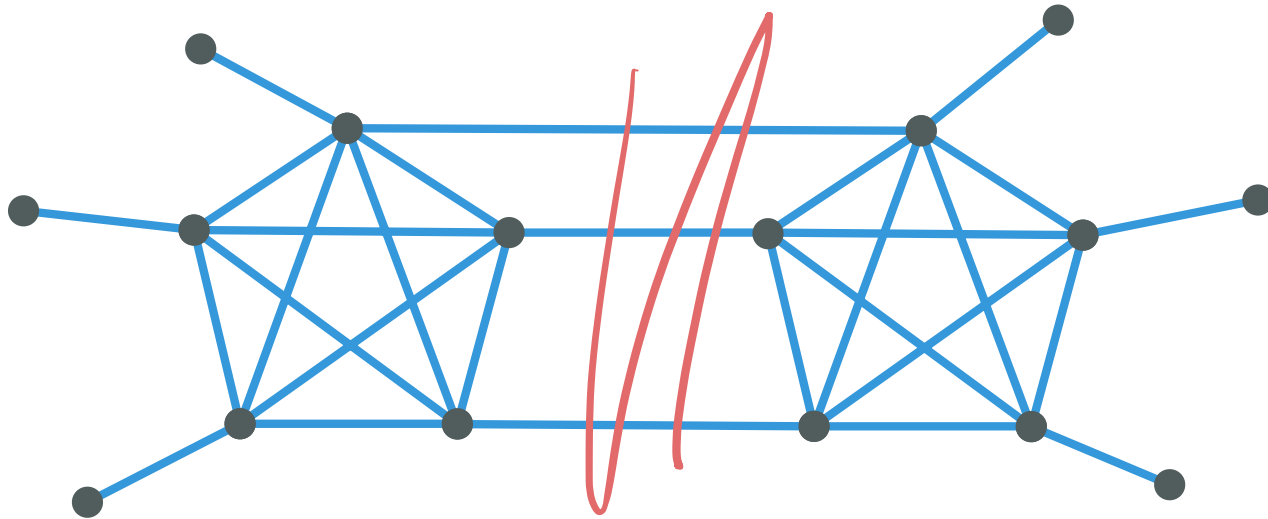
" $\delta(S)$ " = edges cut by S

Recall

Special case: uniform sparsest cut

w/ uniform demands $b(u,v)=1 \quad \forall u,v$

$$\text{"sparsity"} = \frac{c(\delta(S))}{|S| |S^c|}$$



useful primitive for graph partitioning, divide + conquer

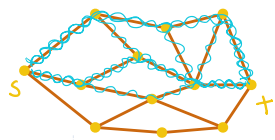
Lecture 10

Probability (and techniques)

LP duality

L_1 -metric embeddings

Max Flow Min Cut



Path Packing LP

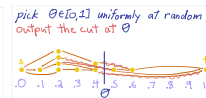
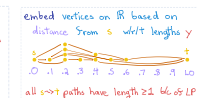
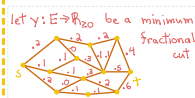
maximize $\sum_{p \in P} x_p$
over $x: P_{st} \rightarrow \mathbb{R}_{\geq 0}$
(all s-t paths)

s.t. $\sum_{p \in P} x_p \leq c_e$ for all edges e
"Ax ≤ c" (capacity of e)

Path Covering LP (i.e. cut)

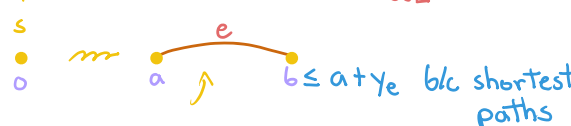
minimize $\sum_{e \in E} c_e y_e$
over $y: E \rightarrow \mathbb{R}_{\geq 0}$
(total capacity of "fractional cut")

s.t. $\sum_{e \in P} y_e \geq 1$ for all $p \in P_{st}$
"A'y ≥ b" (continuous analog of s-t cut)



$E[\text{capacity of cut}]$

$$\stackrel{\text{(linearity of expectation)}}{=} \sum_{e \in E} c_e \Pr[e \text{ cut}] \leq \sum_{e \in E} c_e y_e$$



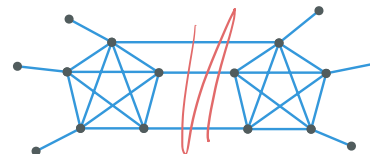
minimum capacity of any fractional cut

Algorithms

Sparsest Cut

Special case: uniform sparsest cut
w/ uniform demands $b(u,v) = 1 \forall u,v$

$$\text{"sparsity"} = \frac{c(\delta(s))}{|S| |T|}$$



useful primitive for graph partitioning, divide & conquer

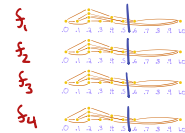
Q: how to round LP to sparse cut?

min $\sum_{e \in E} c_e d(u,v)$ over metrics $d: V \times V \rightarrow \mathbb{R}_{\geq 0}$
s.t. $\sum_{i=1}^k b_i d(s_i, t_i) \geq 1$

Suppose $d(u,v)$ was of the form

$$d(u,v) = \sum_{i=1}^k |f_i(u) - f_i(v)| \text{ for } f_i: V \rightarrow [0,1]^k$$

(assume $\min f_i(a) = 0, \max f_i(a) = 1$)



" L_1 -metric"

Setup / notation

undirected graph $G=(V,E)$

edge weights $w: E \rightarrow \mathbb{R}_{>0}$

for $S \subset V$, " $\delta(S)$ " = set of edges cut by S

" $w(\delta(S))$ " = weight of edges cut by S

for $S \subset V$, " $\text{vol}(S)$ " = $\sum_{v \in S} \deg(v)$

$\bar{S} = V \setminus S$

Conductance

$G=(V,E)$ undirected
edge weights $w: E \rightarrow \mathbb{R}_{\geq 0}$

" $\delta(S)$ " = edges cut by S

" $\text{vol}(S)$ " = $\sum_{v \in S} \deg(v)$

"conductance of S "

$$\text{for } S \subset V, \quad \psi(S) = \frac{w(\delta(S))}{\min\{\text{vol}(S), \text{vol}(\bar{S})\}}$$

sum of
degrees

"conductance of G "

$$\psi(G) = \min_{\emptyset \subsetneq S \subsetneq V} \psi(S)$$

Sparsity

min
S

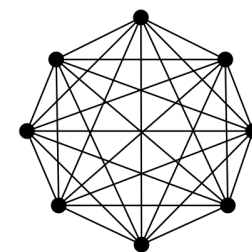
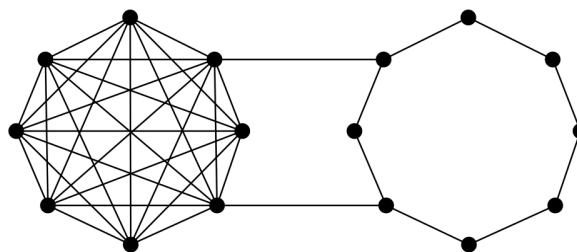
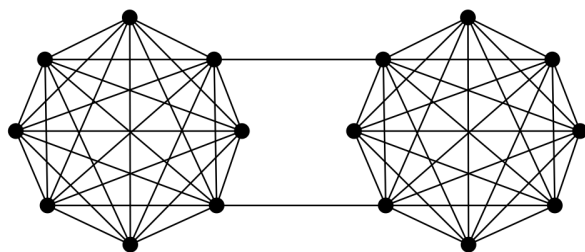
$$\frac{w(\delta(S))}{\min\{|S|, |\bar{S}|\}}$$

Conductance

min
S

$$\frac{w(\delta(S))}{\min\{\text{vol}(S), \text{vol}(\bar{S})\}}$$

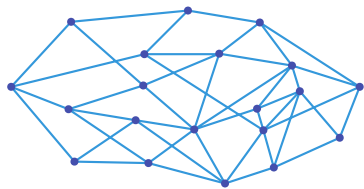
(high/medium/low)



Sparsity

conductance

Laplacian of



$G=(V,E)$ w/ edge weights $w \in \mathbb{R}_{\geq 0}^E$

$$L = \sum_{e \in E} w(e) L_e$$

$$\langle x, Lx \rangle = \sum_{\substack{e \in E \\ e=\{u,v\}}} w(e) (x_u - x_v)^2$$

Rank-1 matrix
 $u \otimes v$ where $u, v \in \mathbb{R}^n$
 $\langle x, (u \otimes v) y \rangle = \langle x, u \rangle \langle v, y \rangle$

Laplacian of $e = \{u, v\}$

$$L_e = (1_u - 1_v) \otimes (1_u - 1_v) \in \mathbb{R}^{n \times n}$$

($1_u = \{0,1\}$ -indicator vector for u)

$$\langle x, L_e x \rangle = (x_u - x_v)^2$$

if $\text{vol}(S) \leq \text{vol}(\bar{S})$,

$$\psi(S) = \frac{w(\delta(S))}{\text{vol}(S)} = \frac{\langle \mathbf{1}_S, L \mathbf{1}_S \rangle}{\langle \mathbf{1}_S, D \mathbf{1}_S \rangle}$$

$G=(V,E)$ undirected
edge weights $w:E \rightarrow \mathbb{R}_{\geq 0}$

" $\delta(S)$ " = edges cut by S

" $\text{vol}(S)$ " = $\sum_{v \in S} \deg(v)$

$$\psi(S) = \frac{w(\delta(S))}{\min\{\text{vol}(S), \text{vol}(\bar{S})\}}$$

$$\psi(G) = \min_{\emptyset \subset S \subset V} \psi(S)$$

Laplacian of 
 $L = \sum_{e \in E} w(e) L_e$

$$\langle x, Lx \rangle = \sum_{e=\{u,v\}} w(e) (x_u - x_v)^2$$

$$D = \text{diag}(\text{degrees})$$

Lemma: $L \in \mathbb{R}^{n \times n}$ symmetric,
 $x \in \mathbb{R}^n$ max/min $\langle x, Lx \rangle$ over $\|x\|=1$
 $\Rightarrow Lx = \mu x$ for $\mu = \langle x, Lx \rangle$

Spectral theorem. $L \in \mathbb{R}^{n \times n}$ symmetric
 $L = \lambda_1 (v_1 \otimes v_1) + \dots + \lambda_n (v_n \otimes v_n)$
where $\lambda_1, \dots, \lambda_n \in \mathbb{R}$
 $v_1, \dots, v_n$ orthonormal basis

Recall

Normalized random walk matrix

$$Q = D^{-1/2} R D^{1/2} = D^{-1/2} A D^{-1/2}$$

- $D = \text{diag}(d)$ w/ $d = \text{degrees}$
- $A = \text{adjacency matrix}$
- $R = A D^{-1} = \text{random walk matrix}$

Q is symmetric!

Q has same eigenvalues as R !

$$Q = v_1 \otimes v_1 + \mu_2 (v_2 \otimes v_2) + \dots + \mu_n (v_n \otimes v_n)$$

$$\text{w/ } \mu_1 = 1 \geq \mu_2 \geq \dots \geq \mu_n$$

(also eigenvalues of R)

Rank-1 matrix
 $u \otimes v$ where $u, v \in \mathbb{R}^n$
 $\langle x, (u \otimes v) y \rangle = \langle x, u \rangle \langle v, y \rangle$

Lemma: $L \in \mathbb{R}^{n \times n}$ symmetric,
 $x \in \mathbb{R}^n$ maximizes $\langle x, Lx \rangle$ over $\|x\|=1$
 $\Rightarrow Lx = \mu x$ for $\mu = \langle x, Lx \rangle$

Spectral theorem. $L \in \mathbb{R}^{n \times n}$ symmetric
 $L = \lambda_1 (v_1 \otimes v_1) + \dots + \lambda_n (v_n \otimes v_n)$
where $\lambda_1, \dots, \lambda_n \in \mathbb{R}$
 $v_1, \dots, v_n$ orthonormal basis

Theorem: $G=(V,E)$ undirected, $w:E \rightarrow \mathbb{R}_{>0}$
then R has stationary distr. $s \in \Delta_V$ w/

$$s_v = \frac{\deg(v)}{2W} \quad \deg(v) = \sum_{e \in \partial(v)} w(e), \quad W = \sum_{e \in E} w(e)$$

Lemma let $A = C^{-1} B C$.

- A and B have same eigenvalues
- C maps eigenvectors of A to eigenvectors of B

Recall

$$Q = \frac{\sqrt{d} \otimes \sqrt{d}}{\langle \mathbf{1}, d \rangle} + \mu_2 (v_2 \otimes v_2) + \dots + \mu_n (v_n \otimes v_n)$$

w/ $\mu_1 = 1 \geq \mu_2 \geq \dots \geq \mu_n$
(also eigenvalues of R)

Theorem. let $\gamma = 1 - \max\{\mu_2, |\mu_n|\}$

$$\|R^k x - s\| \leq (1 - \gamma)^k \sqrt{\frac{\Delta_{\max}}{\Delta_{\min}}}$$

↑
stationary distribution

Rank-1 matrix
 $u \otimes v$ where $u, v \in \mathbb{R}^n$
 $\langle x, (u \otimes v) y \rangle = \langle x, u \rangle \langle v, y \rangle$

Lemma: $L \in \mathbb{R}^{n \times n}$ symmetric,
 $x \in \mathbb{R}^n$ maximizes $\langle x, Lx \rangle$ over $\|x\|=1$
 $\Rightarrow Lx = \mu x$ for $\mu = \langle x, Lx \rangle$

Spectral theorem. $L \in \mathbb{R}^{n \times n}$ symmetric
 $L = \lambda_1 (v_1 \otimes v_1) + \dots + \lambda_n (v_n \otimes v_n)$
where $\lambda_1, \dots, \lambda_n \in \mathbb{R}$
 $v_1, \dots, v_n$ orthonormal basis

Theorem: $G=(V, E)$ undirected, $w: E \rightarrow \mathbb{R}_{>0}$
then R has stationary distr. $s \in \Delta_V$ w/

$$s_v = \frac{\deg(v)}{2W} \quad \deg(v) = \sum_{e \in \mathcal{E}(v)} w(e), \quad W = \sum_{e \in E} w(e)$$

Lemma let $A = C^{-1}BC$.

- A and B have same eigenvalues
- C maps eigenvectors of A to eigenvectors of B

Normalized random walk matrix

$$Q = D^{-1/2} R D^{1/2} = D^{-1/2} A D^{-1/2}$$

- $D = \text{diag}(d)$ w/ $d = \text{degrees}$
- $A = \text{adjacency matrix}$
- $R = A D^{-1} = \text{random walk matrix}$
- Q is symmetric!
- Q has same eigenvalues as R !

normalized Laplacian $M = D^{-1/2} L D^{-1/2}$

Normalized random walk matrix

$$Q = D^{-1/2} R D^{1/2} = D^{-1/2} A D^{-1/2}$$

- $D = \text{diag}(d)$ w/ $d = \text{degrees}$
- $A = \text{adjacency matrix}$
- $R = A D^{-1} = \text{random walk matrix}$

$M =$

$G = (V, E)$ undirected
edge weights $w: E \rightarrow \mathbb{R}_{>0}$

" $\delta(S)$ " = edges cut by S

$$\text{"vol}(S) = \sum_{v \in S} \deg(v)$$

$$\psi(S) = \frac{w(\delta(S))}{\min\{\text{vol}(S), \text{vol}(\bar{S})\}}$$

$$\psi(G) = \min_{\emptyset \subset S \subset V} \psi(S)$$

Laplacian of 
 $L = \sum_{e \in E} w(e) L_e$

$$\langle x, Lx \rangle = \sum_{e=\{u,v\}} w(e) (x_u - x_v)^2$$

$$D = \text{diag}(\text{degrees})$$

$$L = D - A$$

Lemma: $L \in \mathbb{R}^{n \times n}$ symmetric,
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 $L = \lambda_1 (v_1 \otimes v_1) + \dots + \lambda_n (v_n \otimes v_n)$
where $\lambda_1, \dots, \lambda_n \in \mathbb{R}$
 $v_1, \dots, v_n$ orthonormal basis

if $\text{vol}(S) \leq \text{vol}(\bar{S})$,

$$\psi(S) = \frac{w(\delta(S))}{\text{vol}(S)} = \frac{\langle \mathbf{1}_S, L \mathbf{1}_S \rangle}{\langle \mathbf{1}_S, D \mathbf{1}_S \rangle}$$

normalized Laplacian $M = D^{-1/2} L D^{-1/2}$

$$\checkmark M = I - Q = D^{-1/2} (I - R) D^{1/2}$$

Theorem: M has same eigenvalues as $I - R$.

λ for $M \rightarrow$ for R
eigenvalue eigenvalue

λ for $R \rightarrow$ for M
eigenvalue eigenvalue

$G = (V, E)$ undirected
edge weights $w: E \rightarrow \mathbb{R}_{\geq 0}$

" $\delta(S)$ " = edges cut by S

$$\text{"vol}(S) = \sum_{v \in S} \deg(v)$$

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Laplacian of 
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Cheeger's inequality:

let λ_2 = 2nd smallest eigenvalue of M

then
$$\frac{\lambda_2}{2} \leq \Psi(G) \leq \sqrt{2\lambda_2}$$

$$\Psi(S) = \frac{w(S(S))}{\min\{\text{vol}(S), \text{vol}(\bar{S})\}}$$

$$\Psi(G) = \min_{\emptyset \subset S \subset V} \Psi(S)$$

Laplacian of 

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$$D = \text{diag}(\text{degrees})$$

$$L = D - A$$

normalized Laplacian

$$M = D^{-1/2} L D^{-1/2}$$

Normalized random walk matrix

$$Q = D^{-1/2} R D^{1/2} = D^{-1/2} A D^{-1/2}$$

- $D = \text{diag}(d)$ w/ $d = \text{degrees}$
- $A = \text{adjacency matrix}$
- $R = A D^{-1} = \text{random walk matrix}$

Theorem: M has same eigenvalues as $I - R$.

$$\lambda \text{ for } M \rightarrow \lambda - 1 \text{ for } R$$

eigenvalue eigenvalue

$$\lambda \text{ for } R \rightarrow 1 - \lambda \text{ for } M$$

eigenvalue eigenvalue

1

Theorem. let $\gamma = 1 - \max\{\mu_2, |\mu_n|\}$

$$\|R^k x - s\| \leq (1 - \gamma)^k \sqrt{\frac{\Delta_{\max}}{\Delta_{\min}}}$$

↑
stationary
distribution

$1 \geq \mu_2 \geq \dots \geq \mu_n = \text{eigenvalues of } R$

2

Theorem: M has same eigenvalues as $I - R$.

λ for $M \rightarrow \lambda - 1$ for R
eigenvalue eigenvalue

λ for $R \rightarrow 1 - \lambda$ for M
eigenvalue eigenvalue

3

Cheeger's inequality:

$$\frac{\lambda_2}{2} \leq \psi(G) \leq \sqrt{2\lambda_2}$$

$\lambda_2 = 2\text{nd smallest eigenvalue of } M$

let $\Psi = \Psi(G)$

Lemma: if $\mu_2 \geq |\mu_n|$, then

$$\|R^t x - s\| \leq e^{-t\Psi/2} \sqrt{\frac{\Delta_{\max}}{\Delta_{\min}}}$$

↑
stationary
distribution

proof: $1 - \gamma = \square$

Theorem. let $\gamma = 1 - \max\{\mu_2, |\mu_n|\}$

$$\|R^k x - s\| \leq (1 - \gamma)^k \sqrt{\frac{\Delta_{\max}}{\Delta_{\min}}}$$

↑
stationary
distribution

$1 \geq \mu_2 \geq \dots \geq \mu_n = \text{eigenvalues of } R$

Theorem: M has same eigenvalues as $I - R$.

λ for $M \rightarrow \lambda - 1$ for R
eigenvalue eigenvalue

λ for $R \rightarrow 1 - \lambda$ for M
eigenvalue eigenvalue

Cheeger's inequality:

$$\frac{\lambda_2}{2} \leq \Psi(G) \leq \sqrt{2\lambda_2}$$

$\lambda_2 = 2\text{nd smallest eigenvalue of } M$

let $\Psi = \Psi(G)$

✓ Lemma: if $\mu_2 \geq |\mu_n|$, then

$$\|R^t x - s\| \leq e^{-t\Psi/2} \sqrt{\frac{\Delta_{\max}}{\Delta_{\min}}}$$

↑
stationary
distribution

Q:

Theorem. let $\gamma = 1 - \max\{\mu_2, |\mu_n|\}$

$$\|R^k x - s\| \leq (1 - \gamma)^k \sqrt{\frac{\Delta_{\max}}{\Delta_{\min}}}$$

↑
stationary
distribution

$1 \geq \mu_2 \geq \dots \geq \mu_n = \text{eigenvalues of } R$

Theorem: M has same eigenvalues as $I - R$.

λ for $M \rightarrow \lambda - 1$ for R
eigenvalue eigenvalue

λ for $R \rightarrow 1 - \lambda$ for M
eigenvalue eigenvalue

Cheeger's inequality:

$$\frac{\lambda_2}{2} \leq \Psi(G) \leq \sqrt{2\lambda_2}$$

$\lambda_2 = 2\text{nd smallest eigenvalue of } M$

Lazy random walk

w/ prob. $\frac{1}{2}$:



let $\Psi = \Psi(G)$

Theorem:

$$\|S^\dagger x - s\| \leq e^{-\Psi^2/4} \sqrt{\frac{\Delta_{\max}}{\Delta_{\min}}}$$

↑
stationary
distribution

proof:

Theorem. let $\gamma = 1 - \max\{\mu_2, |\mu_n|\}$

$$\|R^k x - s\| \leq (1 - \gamma)^k \sqrt{\frac{\Delta_{\max}}{\Delta_{\min}}}$$

↑
stationary
distribution

$1 \geq \mu_2 \geq \dots \geq \mu_n = \text{eigenvalues of } R$

Theorem: M has same eigenvalues as $I - R$.

λ for $M \rightarrow \lambda - 1$ for R
eigenvalue eigenvalue

λ for $R \rightarrow 1 - \lambda$ for M
eigenvalue eigenvalue

Cheeger's inequality:

$$\frac{\lambda_2}{2} \leq \Psi(G) \leq \sqrt{2\lambda_2}$$

$\lambda_2 = 2\text{nd smallest eigenvalue of } M$

$$S = \frac{1}{2}I + \frac{1}{2}R$$

Cheeger's inequality:

let λ_2 = 2nd smallest eigenvalue of M
then

$$\frac{\lambda_2}{2} \leq \Psi(G) \leq \sqrt{2\lambda_2}$$

- connects combinatorial properties of G w/ algebraic properties

$$\Psi(S) = \frac{w(S(S))}{\min\{\text{vol}(S), \text{vol}(\bar{S})\}}$$

$$\Psi(G) = \min_{\emptyset \subsetneq S \subsetneq V} \Psi(S)$$

Laplacian of 
 $L = \sum_{e \in E} w(e) L_e$

$$\langle x, Lx \rangle = \sum_{e=\{u,v\}} w(e) (x_u - x_v)^2$$

$D = \text{diag}(\text{degrees})$

$$L = D - A$$

normalized Laplacian
 $M = D^{-1/2} L D^{-1/2}$

$$\Psi(G) \leq \sqrt{2\lambda_2}$$

high level ideas:

Theorem: M has same
eigenvalues as $I-R$.

λ for $M \rightarrow \lambda - 1$ for R
eigenvalue eigenvalue

λ for $R \rightarrow 1 - \lambda$ for M
eigenvalue eigenvalue

$$\Psi(G) \leq \sqrt{2\lambda_2}$$

$$M = (\lambda_1=0)(v_1 \otimes v_1) + \lambda_2(v_2 \otimes v_2) + \dots + \lambda_n(v_n \otimes v_n)$$

$v_1, \dots, v_n$ orthonormal, $\lambda_1=0 \leq \lambda_2 \leq \dots \leq \lambda_n$

v_1 = eigenvector for eigenvalue 1 of R
 $= \sqrt{d}/\langle 1, d \rangle$

Theorem: M has same eigenvalues as $I-R$.
 λ for $M \rightarrow \lambda-1$ for R
 λ for $R \rightarrow 1-\lambda$ for M

high level ideas:

$$\lambda_2 = \langle v_2, M v_2 \rangle = \min \frac{\langle y, M y \rangle}{\langle y, y \rangle} \text{ s.t. } \langle y, \sqrt{d} \rangle = 0$$

$$= \min \frac{\langle x, L x \rangle}{\langle x, D x \rangle} \text{ s.t. } \langle x, d \rangle = 0$$

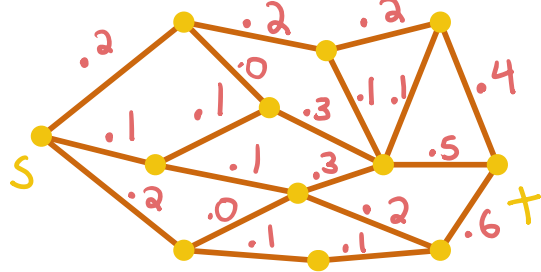
goal: convert x to $S \subseteq V$ s.t.

$$\Psi(S) \leq \sqrt{\frac{\langle x, L x \rangle}{\langle x, D x \rangle}} = \sqrt{\lambda_2}$$

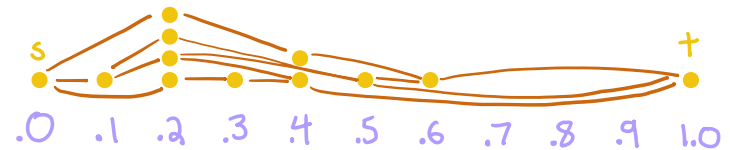
how?

Recall:

let $\gamma: E \rightarrow \mathbb{R}_{\geq 0}$ be a minimum fractional cut



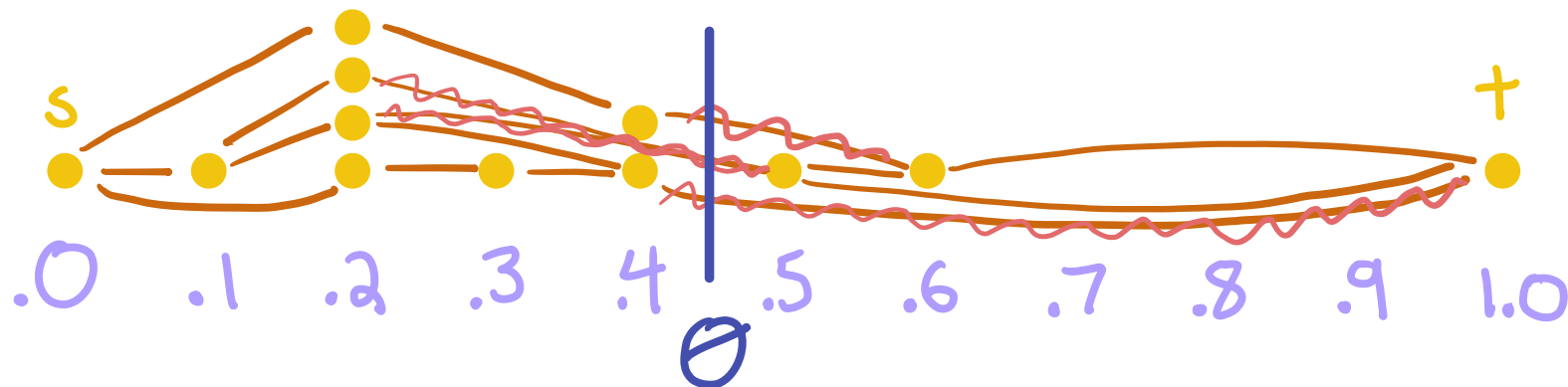
embed vertices on $\mathbb{R}$ based on distance from s w/r/t lengths γ



all $s \rightsquigarrow t$ paths have length ≥ 1 b/c of LP

pick $\theta \in [0, 1]$ uniformly at random

output the cut at θ



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high level ideas:

$$\lambda_2 = \langle v_2, M v_2 \rangle = \min \frac{\langle x, Lx \rangle}{\langle x, Dx \rangle} \text{ s.t. } \langle x, d \rangle = 0$$

claim: if $\langle x, d \rangle = 0$, line embedding of x induces

$$\text{cut } S \text{ s.t. } \Psi(S) \leq \sqrt{2 \frac{\langle x, Lx \rangle}{\langle x, Dx \rangle}}$$

let $V = \sum_v \deg(v) = \text{total volume}$

Lemma: let $x \in \mathbb{R}_{\geq 0}^V$ s.t. $\text{vol}(\text{support}(x)) \leq \frac{V}{2}$

then $\langle x, Lx \rangle \geq \frac{\psi^2}{2} \langle x, Dx \rangle$

$$\psi(S) = \frac{w(S(S))}{\min\{\text{vol}(S), \text{vol}(\bar{S})\}}$$

$$\psi(G) = \min_{\emptyset \subsetneq S \subsetneq V} \psi(S)$$

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Proof.

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We want:

$$(\gamma = D^{\frac{1}{2}} x)$$

$$\frac{1}{2} \psi^2 \leq \lambda_2 = \min_{\langle \sqrt{d}, \gamma \rangle = 0} \frac{\langle \gamma, M \gamma \rangle}{\langle \gamma, \gamma \rangle} = \min_{\langle d, x \rangle = 0} \frac{\langle x, L x \rangle}{\langle x, D x \rangle}$$

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Claims:

•

•

$$\bullet \frac{\langle x+d\mathbb{1}, L(x+d\mathbb{1}) \rangle}{\langle x+d\mathbb{1}, D(x+d\mathbb{1}) \rangle} \leq \frac{\langle x, Lx \rangle}{\langle x, Dx \rangle} \quad \forall d \in \mathbb{R}$$

$$\bullet \begin{aligned} \langle x, Lx \rangle &\geq \langle x_+, Lx_+ \rangle + \langle x_-, Lx_- \rangle \\ \langle x, Dx \rangle &\leq \langle x_+, Dx_+ \rangle + \langle x_-, Dx_- \rangle \end{aligned} \quad \text{w/} \quad \begin{array}{l} \text{positive} \quad \text{negative} \\ \text{part} \quad \text{part} \\ x = x_+ - x_- \\ x_+, x_- \geq 0 \end{array}$$

1.

2.

3.

1. let x attain $\min_{\langle d, x \rangle = 0} \frac{\langle x, Lx \rangle}{\langle x, Dx \rangle} = \lambda_2$

2. $x \leftarrow x + d\mathbb{1}$ w/ $\text{vol}(\{v: x_v > d\}), \text{vol}(\{v: x_v < d\}) \leq \frac{V}{2}$

3. $\min \left\{ \frac{\langle x_+, Lx_+ \rangle}{\langle x_+, Dx_+ \rangle}, \frac{\langle x_-, Lx_- \rangle}{\langle x_-, Dx_- \rangle} \right\} \leq \frac{\langle x_+, Lx_+ \rangle + \langle x_-, Lx_- \rangle}{\langle x_+, Dx_+ \rangle + \langle x_-, Dx_- \rangle} \leq \frac{\langle x, Lx \rangle}{\langle x, Dx \rangle} = \lambda_2$

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Lemma: let $\langle x, d \rangle = 0$. then $\forall y = x + d \mathbb{1}$,

$$\frac{\langle y, Ly \rangle}{\langle y, Dy \rangle} \leq \frac{\langle x, Lx \rangle}{\langle x, Dx \rangle}$$

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$$\langle x, Lx \rangle = \langle x_+, Lx_+ \rangle + \langle x_-, Lx_- \rangle - 2 \langle x_+, Lx_- \rangle$$

$$\langle x_+, Lx_- \rangle = \sum_{e=uv} w_e (x_+(u) - x_+(v))(x_-(u) - x_-(v))$$

claim: $\forall a, b, \underbrace{(a^+ - b^+)}_{\geq 0} \underbrace{(a^- - b^-)}_{\leq 0} \leq 0$

why? WLOG $a \geq b$.

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4. Lemma $\Rightarrow$

$$\frac{\psi^2}{2} \leq \min \left\{ \frac{\langle x_+, Lx_+ \rangle}{\langle x_+, Dx_+ \rangle}, \frac{\langle x_-, Lx_- \rangle}{\langle x_-, Dx_- \rangle} \right\}$$

$$\leq \lambda_2$$

Cheeger's inequality:

let λ_2 = 2nd smallest eigenvalue of M
then

exercise

$$\frac{\lambda_2}{2} \leq \underbrace{\Psi(G)}_{\text{algorithmic, rest of class}} \leq \sqrt{2\lambda_2}$$

algorithmic, rest of class

- connects combinatorial properties of G w/ algebraic properties

$$\Psi(S) = \frac{w(S)}{\min\{\text{vol}(S), \text{vol}(\bar{S})\}}$$

$$\Psi(G) = \min_{\emptyset \subsetneq S \subsetneq V} \Psi(S)$$

Laplacian of 

$$L = \sum_{e \in E} w(e) L_e$$

$$\langle x, Lx \rangle = \sum_{e=\{u,v\}} w(e) (x_u - x_v)^2$$

$$D = \text{diag}(\text{degrees})$$

$$L = D - A$$

normalized Laplacian

$$M = D^{-1/2} L D^{-1/2}$$