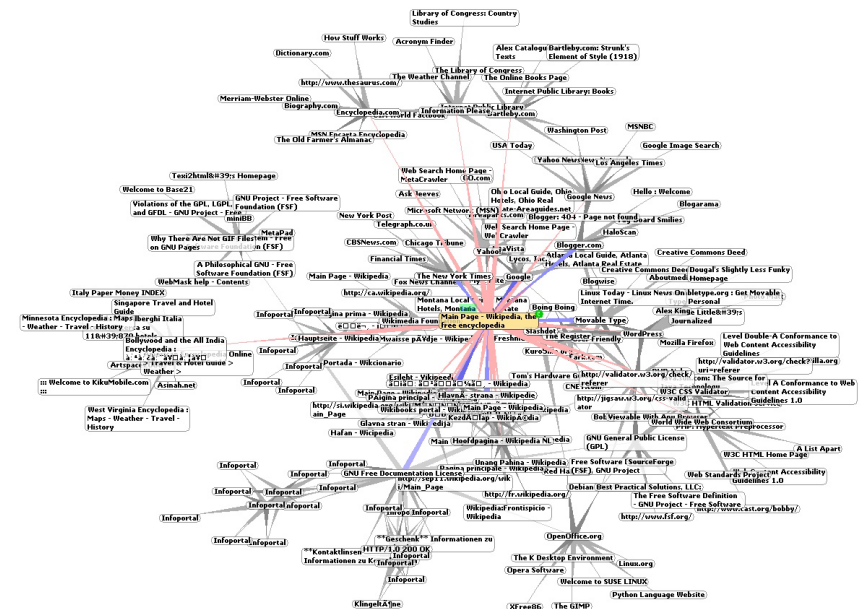


# Spectral analysis of undirected random walks

# Perron-Frobenius Theorem



# Lecture 18

## Probability (and techniques)

## Algorithms (models and applications)

Hitting times

Cover times

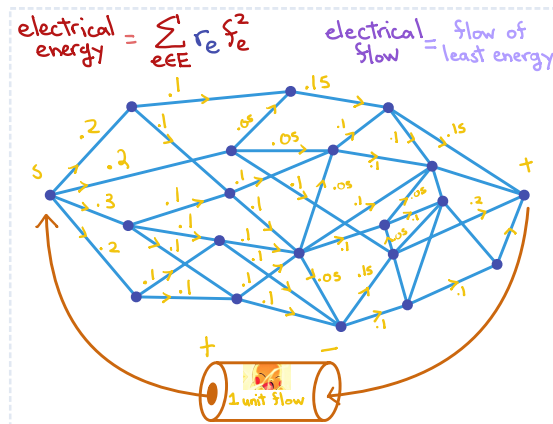
Electrical networks

Connectivity  
in  $O(\log n)$  space

random walk terminology:

Hitting time  $H(s, t)$  = expected # steps until a random walk from  $s$  reaches  $t$

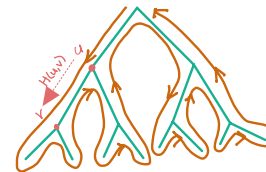
Cover time  $C(s)$  = expected # steps until a random walk from  $s$  visits every vertex



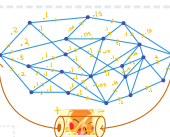
Lemma  $H(u, v) + H(v, u) = 2m \cdot (\text{energy of unit flow } (u, v))$

Theorem  $C(s) \leq 2mn$

proof Six traversal of spanning tree



$$\begin{aligned} \text{total time} &= \sum_{u, v \in V} H(u, v) + H(v, u) \\ &= 2m \sum_{u, v \in V} (\text{energy of unit flow } (u, v)) \leq 2m(n-1) \end{aligned}$$



electrical energy =  $\sum_{e \in E} r_e f_e^2$   
electrical flow = least energy

des  $B: \mathbb{R}^E \rightarrow \mathbb{R}^V$  by  
(BS) $_v$  = net flow at  $v$   
 $d_v = \begin{cases} -1 & \text{if } v = s \\ 1 & \text{if } v = t \\ 0 & \text{otherwise} \end{cases}$

min  $\frac{1}{2} \sum_{e \in E} r_e f_e^2$  s.t.  $BS = d$   
(electrical energy) (is a typical)

effective resistance of  $d$  = optimum value/energy  
Ohm's law:  $f = R^{-1} B^T p$   
for electric flow by electric potentials  $p$   
 $L = BR^{-1}B^T \in \mathbb{R}^{V \times V}$   
(effective resist. of  $d$ )  $= \langle d, L^{-1}d \rangle$

# Lecture 19

Probability (and techniques)

Algorithms (models and applications)

Spectral theorem  
for symmetric maps

Spectral analysis of  
random walks



# Recall:

$$\text{score}_3(v) = \sum_{(u,v) \in E} \frac{\text{score}_3(u)}{\deg^+(u)}$$



let  $\Delta^V = \{x \in \mathbb{R}_V^{\geq 0} : \sum_v x_v = 1\} = \{\text{distributions over } V\}$

Theorem there exists  $x \in \Delta^V$  s.t.

$$x_v = \sum_{(u,v) \in E} \frac{x_u}{\deg^+(u)} \quad \forall v \in V.$$

If  $G$  is strongly connected, then  $x$  is unique and strictly positive.

Theorem, restated: every random walk has a stationary distribution. If  $G$  is strongly connected, then this distribution is unique.

Theorem, restated: let  $G$  be a dir. graph and  $R$  the random walk linear map. Then  $R$  has an eigenvector  $x \in \Delta^V$  w/ eigenvalue 1. If  $G$  is strongly connected, then  $x$  is the unique eigenvector w/ eigenvalue 1

Today, for undirected graphs

random walk

Let  $R$  have eigenvalues

$$\mu_1 = 1 \geq \mu_2 \geq \dots \geq \mu_n \geq -1$$

Theorem. let  $\gamma = 1 - \max\{\mu_2, |\mu_n|\}$

$$\|R^k x - s\| \leq (1 - \gamma)^k \sqrt{\frac{\Delta_{\max}}{\Delta_{\min}}}$$

↑  
stationary  
distribution

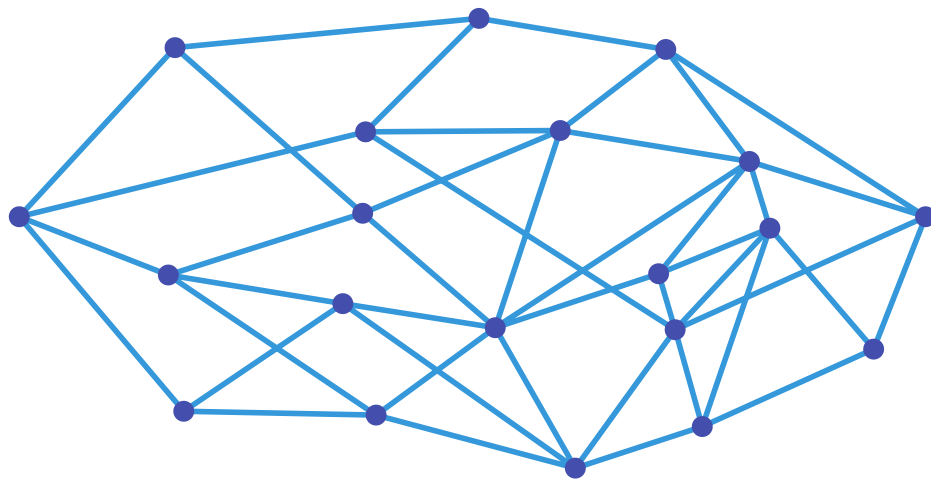
# Lecture 19

Probability (and techniques)

Algorithms (models and applications)

Spectral theorem  
for symmetric maps

Spectral analysis of  
random walks



Laplacian

$$L \in \mathbb{R}^{n \times n}$$

(symmetric, positive semidefinite)

# Rank-1 matrix

$$u \otimes v \quad \text{where } u, v \in \mathbb{R}^n$$

(aka.  $uv^T$ )

defined by  $\langle x, (u \otimes v) y \rangle = \langle x, u \rangle \langle v, y \rangle$

as a matrix:

$$u \otimes v = \begin{bmatrix} u_1 v_1 & u_1 v_2 & \dots & u_1 v_n \\ u_2 v_1 & u_2 v_2 & \dots & u_2 v_n \\ \vdots & \vdots & \ddots & \vdots \\ u_n v_1 & u_n v_2 & \dots & u_n v_n \end{bmatrix}$$

e.g.

$$A(u \otimes v)B = (Au) \otimes (Bv)$$

$$(u \otimes v)(x \otimes y) = \langle v, x \rangle (u \otimes y)$$

Laplacian of an edge  $e = \{u, v\}$

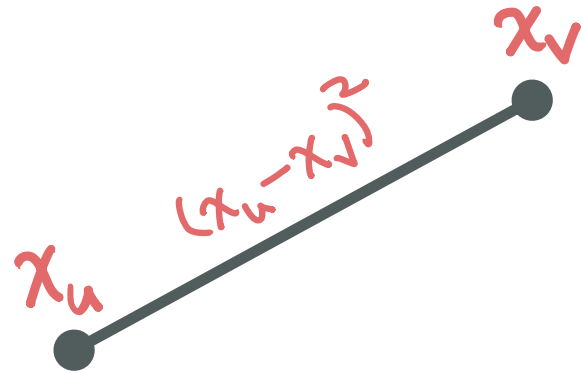
Rank-1 matrix  
 $u \otimes v$  where  $u, v \in \mathbb{R}^n$   
 $\langle x, (u \otimes v) y \rangle = \langle x, u \rangle \langle v, y \rangle$

$$L_e = (1_u - 1_v) \otimes (1_u - 1_v) \in \mathbb{R}^{V \times V}$$

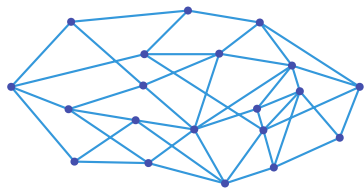
$(1_u = \{0, 1\}$ -indicator vector for  $u$ )  
 $(0, \dots, 0, \underbrace{1}_{u\text{th coord}}, 0, \dots, 0)$

$$\langle x, L_e x \rangle = (x_u - x_v)^2$$

$$(\langle 1_u - 1_v, x \rangle = x_u - x_v)$$



Laplacian of



$G=(V,E)$  w/ edge weights  $w \in \mathbb{R}_{\geq 0}^E$

$$L = \sum_{e \in E} w(e) L_e$$

$$\langle x, Lx \rangle = \sum_{\substack{e \in E \\ e=\{u,v\}}} w(e) (x_u - x_v)^2$$

Rank-1 matrix

$u \otimes v$  where  $u, v \in \mathbb{R}^n$

$$\langle x, (u \otimes v) y \rangle = \langle x, u \rangle \langle v, y \rangle$$

Laplacian of  $e = \{u, v\}$

$$L_e = (1_u - 1_v) \otimes (1_u - 1_v) \in \mathbb{R}^{V \times V}$$

( $1_u = \{0,1\}$ -indicator vector for  $u$ )

$$\langle x, L_e x \rangle = (x_u - x_v)^2$$

$$L = \sum_{e \in E} w(e) L_e$$

$$\langle x, Lx \rangle = \sum_{\substack{e \in E \\ e = \{u, v\}}} w(e) (x_u - x_v)^2$$

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 $u \otimes v$  where  $u, v \in \mathbb{R}^n$   
 $\langle x, (u \otimes v) y \rangle = \langle x, u \rangle \langle v, y \rangle$

Laplacian of  $e = \{u, v\}$

$$L_e = (1_u - 1_v) \otimes (1_u - 1_v) \in \mathbb{R}^{V \times V}$$

( $1_u = \{0, 1\}$ -indicator vector for  $u$ )

$$\langle x, L_e x \rangle = (x_u - x_v)^2$$

$A \in \mathbb{R}^{n \times n}$  is positive semi-definite if:

- $A$  is symmetric
- $\langle x, Ax \rangle \geq 0 \quad \forall x$

( $A$  is positive definite if it is positive semi-definite & invertible)



Recall:

$$Ax = \lambda x$$

"eigenvalue" (pointing to  $\lambda$ )

"eigenvector" (pointing to  $x$ )

Lemma every linear map  $A$  has an eigenvalue  $\lambda \in \mathbb{C}$  and eigenvector  $x \in \mathbb{C}^n$

Rank-1 matrix  
 $u \otimes v$  where  $u, v \in \mathbb{R}^n$   
 $\langle x, (u \otimes v) y \rangle = \langle x, u \rangle \langle v, y \rangle$

Laplacian of  $e = \{u, v\}$

$$L_e = (1_u - 1_v) \otimes (1_u - 1_v) \in \mathbb{R}^{n \times n}$$

( $1_u = \{0, 1\}$ -indicator vector for  $u$ )

$$\langle x, L_e x \rangle = (x_u - x_v)^2$$

Laplacian of 

$$L = \sum_{e \in E} w(e) L_e$$

$$\langle x, Lx \rangle = \sum_{e = \{u, v\}} w(e) (x_u - x_v)^2$$

positive semi-definite  $A$ :

- $A$  is symmetric
- $\langle x, Ax \rangle \geq 0 \quad \forall x$

Lemma every linear map  $A$  has an eigenvalue  $\lambda \in \mathbb{C}$  and eigenvector  $x \in \mathbb{C}^n$

Lemma:  $L \in \mathbb{R}^{n \times n}$  symmetric,

$x \in \mathbb{R}^n$  maximizes  $\langle x, Lx \rangle$  over  $\|x\|=1$

$\Rightarrow Lx = \mu x$  for  $\mu = \langle x, Lx \rangle$

note:  $\mu = \langle x, Lx \rangle \in \mathbb{R}$

Rank-1 matrix  
 $u \otimes v$  where  $u, v \in \mathbb{R}^n$   
 $\langle x, (u \otimes v) y \rangle = \langle x, u \rangle \langle v, y \rangle$

Laplacian of  $e = \{u, v\}$

$$L_e = (1_u - 1_v) \otimes (1_u - 1_v) \in \mathbb{R}^{n \times n}$$

( $1_u = \{0, 1\}$ -indicator vector for  $u$ )

$$\langle x, L_e x \rangle = (x_u - x_v)^2$$

Laplacian of 

$$L = \sum_{e \in E} w(e) L_e$$

$$\langle x, Lx \rangle = \sum_{e = \{u, v\}} w(e) (x_u - x_v)^2$$

positive semi-definite  $A$ :

- $A$  is symmetric
- $\langle x, Ax \rangle \geq 0 \quad \forall x$

"eigenvalue"

$$Ax = \lambda x$$

"eigenvector"

Lemma:  $L \in \mathbb{R}^{n \times n}$  symmetric,  
 $x \in \mathbb{R}^n$  maximizes  $\langle x, Lx \rangle$  over  $\|x\|=1$   
 $\Rightarrow Lx = \mu x$  for  $\mu = \langle x, Lx \rangle$

proof.

claim: for  $u \in \mathbb{R}^n$ ,

if  $\langle u, x \rangle = 0$ , then  $\langle u, Lx \rangle = 0$ .

if claim is true:

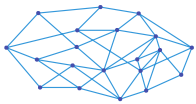
$$\Rightarrow Lx \in \text{span}(x)$$

$$\Rightarrow Lx = \mu_1 x \text{ for some } \mu_1 \in \mathbb{R}$$

$$\Rightarrow \mu_1 = \mu_1 \langle x, x \rangle = \langle x, Lx \rangle$$

Rank-1 matrix  
 $u \otimes v$  where  $u, v \in \mathbb{R}^n$   
 $\langle x, (u \otimes v) y \rangle = \langle x, u \rangle \langle v, y \rangle$

Laplacian of  $e = \{u, v\}$   
 $L_e = (1_u - 1_v) \otimes (1_u - 1_v) \in \mathbb{R}^{n \times n}$   
 $(1_u = \{0, 1\}$ -indicator vector for  $u$ )  
 $\langle x, L_e x \rangle = (x_u - x_v)^2$

Laplacian of   
 $L = \sum_{e \in E} w(e) L_e$   
 $\langle x, Lx \rangle = \sum_{e = \{u, v\}} w(e) (x_u - x_v)^2$

positive semi-definite  $A$ :

- $A$  is symmetric
- $\langle x, Ax \rangle \geq 0 \quad \forall x$

"eigenvalue"  
 $Ax = \lambda x$   
 "eigenvector"

$L \in \mathbb{R}^{n \times n}$  symmetric,  
 $x \in \mathbb{R}^n$  maximizes  $\langle x, Lx \rangle$  over  $\|x\|=1$

claim: if  $\langle u, x \rangle = 0$ , then  $\langle u, Lx \rangle = 0$ .

let  $u \in \mathbb{R}^n$  w/  $\langle u, x \rangle = 0$ ,  $\|u\|=1$

define  $f(\varepsilon) = \frac{\langle x + \varepsilon u, L(x + \varepsilon u) \rangle}{1 + \varepsilon^2}$

(note:  $\|x + \varepsilon u\|^2 = \|x\|^2 + \varepsilon^2 \|u\|^2 = 1 + \varepsilon^2$ )

Rank-1 matrix  
 $u \otimes v$  where  $u, v \in \mathbb{R}^n$   
 $\langle x, (u \otimes v) y \rangle = \langle x, u \rangle \langle v, y \rangle$

Laplacian of  $e = \{u, v\}$   
 $L_e = (1_u - 1_v) \otimes (1_u - 1_v) \in \mathbb{R}^{n \times n}$   
 $(1_u = \{0, 1\}$ -indicator vector for  $u$ )  
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Laplacian of   
 $L = \sum_{e \in E} w(e) L_e$   
 $\langle x, Lx \rangle = \sum_{e = \{u, v\}} w(e) (x_u - x_v)^2$

positive semi-definite  $A$ :

- $A$  is symmetric
- $\langle x, Ax \rangle \geq 0 \quad \forall x$

"eigenvalue"  
 $Ax = \lambda x$   
 "eigenvector"

$L \in \mathbb{R}^{n \times n}$  symmetric,  
 $x \in \mathbb{R}^n$  maximizes  $\langle x, Lx \rangle$  over  $\|x\|=1$

claim: if  $\langle u, x \rangle = 0$ , then  $\langle u, Lx \rangle = 0$ .

let  $u \in \mathbb{R}^n$  w/  $\langle u, x \rangle = 0$ ,  $\|u\|=1$

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(note:  $\|x + \varepsilon u\|^2 = \|x\|^2 + \varepsilon^2 \|u\|^2 = 1 + \varepsilon^2$ )

$x$  max.  $\langle x, Lx \rangle$

$\Rightarrow 0$  max.  $f(x)$

$\Rightarrow f'(0) = 0$

$\Rightarrow \langle u, Lx \rangle = 0$

Rank-1 matrix  
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 $\langle x, (u \otimes v) y \rangle = \langle x, u \rangle \langle v, y \rangle$

Laplacian of  $e = \{u, v\}$   
 $L_e = (1_u - 1_v) \otimes (1_u - 1_v) \in \mathbb{R}^{n \times n}$   
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Laplacian of 

$L = \sum_{e \in E} w(e) L_e$

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"eigenvalue"

$Ax = \lambda x$

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Lemma:  $L \in \mathbb{R}^{n \times n}$  symmetric,  
 $x \in \mathbb{R}^n$  maximizes  $\langle x, Lx \rangle$  over  $\|x\|=1$   
 $\Rightarrow Lx = \mu x$  for  $\mu = \langle x, Lx \rangle$

proof.

✓ for  $u \in \mathbb{R}^n$ ,

if  $\langle u, x \rangle = 0$ , then  $\langle u, Lx \rangle = 0$ .

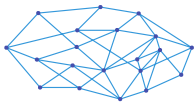
$\Rightarrow Lx \in \text{span}(x)$

$\Rightarrow Lx = \mu_1 x$  for some  $\mu_1 \in \mathbb{R}$

$\Rightarrow \mu_1 = \mu_1 \langle x, x \rangle = \langle x, Lx \rangle$

Rank-1 matrix  
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Laplacian of  $e = \{u, v\}$   
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Laplacian of   
 $L = \sum_{e \in E} w(e) L_e$

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"eigenvalue"  
 $Ax = \lambda x$   
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Lemma:  $L \in \mathbb{R}^{n \times n}$  symmetric,  
 $x \in \mathbb{R}^n$  maximizes  $\langle x, Lx \rangle$  over  $\|x\|=1$   
 $\Rightarrow Lx = \mu x$  for  $\mu = \langle x, Lx \rangle$

Spectral theorem.  $L \in \mathbb{R}^{n \times n}$  symmetric

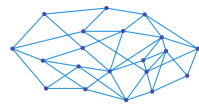
then  $L = \lambda_1 (v_1 \otimes v_1) + \dots + \lambda_n (v_n \otimes v_n)$

where  $\lambda_1, \dots, \lambda_n \in \mathbb{R}$

$v_1, \dots, v_n$  orthonormal basis

Rank-1 matrix  
 $u \otimes v$  where  $u, v \in \mathbb{R}^n$   
 $\langle x, (u \otimes v) y \rangle = \langle x, u \rangle \langle v, y \rangle$

Laplacian of  $e = \{u, v\}$   
 $L_e = (1_u - 1_v) \otimes (1_u - 1_v) \in \mathbb{R}^{n \times n}$   
( $1_u = \{0, 1\}$ -indicator vector for  $u$ )  
 $\langle x, L_e x \rangle = (x_u - x_v)^2$

Laplacian of   
 $L = \sum_{e \in E} w(e) L_e$

$\langle x, Lx \rangle = \sum_{e = \{u, v\}} w(e) (x_u - x_v)^2$

positive semi-definite  $A$ :

- $A$  is symmetric
- $\langle x, Ax \rangle \geq 0 \quad \forall x$

"eigenvalue"  
 $Ax = \lambda x$   
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Spectral theorem.  $L \in \mathbb{R}^{n \times n}$  symmetric

$$\text{then } L = \lambda_1(v_1 \otimes v_1) + \dots + \lambda_n(v_n \otimes v_n)$$

where  $\lambda_1, \dots, \lambda_n \in \mathbb{R}$

$v_1, \dots, v_n$  orthonormal basis

proof by induction on  $n$ .

$n=0$  ✓

$n \geq 1$ : let  $\mu_n \in \mathbb{R}$ ,  $v_n \in \mathbb{R}^n$  be eigenvalue/vector

$$B \stackrel{\text{def}}{=} A - \mu_n(v_n \otimes v_n) \quad Y \stackrel{\text{def}}{=} \{y \in \mathbb{R}^n, \langle v_n, y \rangle = 0\}$$

claim:  $B$  maps  $Y$  to  $Y$

Rank-1 matrix  
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 $\langle x, (u \otimes v) y \rangle = \langle x, u \rangle \langle v, y \rangle$

Laplacian of  $e = \{u, v\}$   
 $L_e = (1_u - 1_v) \otimes (1_u - 1_v) \in \mathbb{R}^{n \times n}$   
( $1_u = \{0, 1\}$ -indicator vector for  $u$ )  
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Laplacian of   
 $L = \sum_{e \in E} w(e) L_e$

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positive semi-definite  $A$ :

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- $\langle x, Ax \rangle \geq 0 \quad \forall x$

Lemma:  $L \in \mathbb{R}^{n \times n}$  symmetric,  
 $x \in \mathbb{R}^n$  maximizes  $\langle x, Lx \rangle$  over  $\|x\|=1$   
 $\Rightarrow Lx = \mu x$  for  $\mu = \langle x, Lx \rangle$



Rank-1 matrix  
 $u \otimes v$  where  $u, v \in \mathbb{R}^n$   
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Laplacian of  $e = \{u, v\}$

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Laplacian of 

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$$\Rightarrow Lx = \mu x \quad \text{for } \mu = \langle x, Lx \rangle$$

Spectral theorem.  $L \in \mathbb{R}^{n \times n}$  symmetric

$$L = \lambda_1 (v_1 \otimes v_1) + \dots + \lambda_n (v_n \otimes v_n)$$

where  $\lambda_1, \dots, \lambda_n \in \mathbb{R}$

$v_1, \dots, v_n$  orthonormal basis

For  $x \in \mathbb{R}^n$ , write

$$x = d_1 v_1 + \dots + d_n v_n \quad (\text{unique, } d_i = \langle x, v_i \rangle)$$

$$Lx = \lambda_1 (v_1 \otimes v_1)x + \dots + \lambda_n (v_n \otimes v_n)x$$

$$= \lambda_1 \langle v_1, x \rangle v_1 + \dots + \lambda_n \langle v_n, x \rangle v_n$$

$$= \lambda_1 d_1 v_1 + \dots + \lambda_n d_n v_n$$

just like a diagonal matrix!

Rank-1 matrix  
 $u \otimes v$  where  $u, v \in \mathbb{R}^n$   
 $\langle x, (u \otimes v)y \rangle = \langle x, u \rangle \langle v, y \rangle$

Laplacian of  $e = \{u, v\}$

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( $1_u = \{0, 1\}$ -indicator vector for  $u$ )

$$\langle x, L_e x \rangle = (x_u - x_v)^2$$

Laplacian of 

$$L = \sum_{e \in E} w(e) L_e$$

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- $A$  is symmetric
- $\langle x, Ax \rangle \geq 0 \quad \forall x$

Lemma:  $L \in \mathbb{R}^{n \times n}$  symmetric,  
 $x \in \mathbb{R}^n$  maximizes  $\langle x, Lx \rangle$  over  $\|x\|=1$   
 $\Rightarrow Lx = \mu x$  for  $\mu = \langle x, Lx \rangle$

# Lecture 19

Probability (and techniques)

Algorithms (models and applications)

✓ Spectral theorem  
for symmetric maps

Spectral analysis of  
random walks

# Recall:

$$\text{score}_3(v) = \sum_{(u,v) \in E} \frac{\text{score}_3(u)}{\deg^+(u)}$$



let  $\Delta^V = \{x \in \mathbb{R}_V^{\geq 0} : \sum_v x_v = 1\} = \{\text{distributions over } V\}$

Theorem there exists  $x \in \Delta^V$  s.t.

$$x_v = \sum_{(u,v) \in E} \frac{x_u}{\deg^+(u)} \quad \forall v \in V.$$

If  $G$  is strongly connected, then  $x$  is unique and strictly positive.

Theorem, restated: every random walk has a stationary distribution. If  $G$  is strongly connected, then this distribution is unique.

Theorem, restated: let  $G$  be a dir. graph and  $R$  the random walk linear map. Then  $R$  has an eigenvector  $x \in \Delta^V$  w/ eigenvalue 1. If  $G$  is strongly connected, then  $x$  is the unique eigenvector w/ eigenvalue 1

Theorem:  $G=(V,E)$  undirected,  $w:E \rightarrow \mathbb{R}_{>0}$

let  $R:\mathbb{R}^V \rightarrow \mathbb{R}^V$  be random walk map

then  $R$  has stationary distr.  $s \in \Delta_V$  w/

$$s_v = \frac{\deg(v)}{2W}$$

$$\deg(v) = \sum_{e \in \partial(v)} w(e)$$

$$W = \sum_{e \in E} w(e)$$

Proof recall  $R = AD^{-1}$ ,  $D = \text{diag}(d)$ ,  
 $d = \text{degrees}$

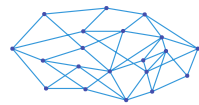
$$Rd = AD^{-1}d = A\mathbb{1} = d$$

$\Rightarrow d$  unique eigenvector w/ eigenvalue 1

$\Rightarrow$  stationary dist. prop. to  $d$

Rank-1 matrix  
 $u \otimes v$  where  $u, v \in \mathbb{R}^n$   
 $\langle x, (u \otimes v) \rangle = \langle x, u \rangle \langle v, x \rangle$

Laplacian of  $e = \{u, v\}$   
 $L_e = (1_u - 1_v) \otimes (1_u - 1_v) \in \mathbb{R}^{V \times V}$   
 $(1_u = \{0,1\}\text{-indicator vector for } u)$   
 $\langle x, L_e x \rangle = (x_u - x_v)^2$

Laplacian of   
 $L = \sum_{e \in E} w(e) L_e$

$$\langle x, Lx \rangle = \sum_{e=\{u,v\}} w(e) (x_u - x_v)^2$$

positive semi-definite  $A$ :

- $A$  is symmetric
- $\langle x, Ax \rangle \geq 0 \quad \forall x$

Lemma:  $L \in \mathbb{R}^{n \times n}$  symmetric,  
 $x \in \mathbb{R}^n$  maximizes  $\langle x, Lx \rangle$  over  $\|x\|=1$   
 $\Rightarrow Lx = \mu x$  for  $\mu = \langle x, Lx \rangle$

Spectral theorem.  $L \in \mathbb{R}^{n \times n}$  symmetric  
 $L = \lambda_1 (v_1 \otimes v_1) + \dots + \lambda_n (v_n \otimes v_n)$   
 where  $\lambda_1, \dots, \lambda_n \in \mathbb{R}$   
 $v_1, \dots, v_n$  orthonormal basis

Let  $R$  have eigenvalues

$$\mu_1 = 1 \geq \mu_2 \geq \dots \geq \mu_n \geq -1$$

Theorem. let  $\gamma = 1 - \max\{\mu_2, |\mu_n|\}$

$$\|R^k x - s\| \leq (1 - \gamma)^k \sqrt{\frac{\Delta_{\max}}{\Delta_{\min}}}$$

↑  
stationary  
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# Similar matrices

$A, B \in \mathbb{R}^{n \times n}$  are similar if

$$A = C^{-1}BC \text{ for some invertible } C \in \mathbb{R}^{n \times n}$$

let  $A = C^{-1}BC$ .

Lemma C restricts to isomorphism  
between  $\ker(A)$  and  $\ker(B)$ .

proof:

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let  $A = C^{-1}BC$ .

Lemma  $C$  restricts to isomorphism between  $\ker(A)$  and  $\ker(B)$ .

Lemma

- $A$  and  $B$  have same eigenvalues
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why?

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$Q$  is symmetric!

$Q$  has same eigenvalues as  $R$ !

Spectral theorem:

$$Q = v_1 \otimes v_1 + \mu_2 (v_2 \otimes v_2) + \dots + \mu_n (v_n \otimes v_n) \quad (v_i \text{ is orthonormal})$$

$$\text{w/ } \mu_1 = 1 \geq \mu_2 \geq \dots \geq \mu_n$$

(also eigenvalues of  $R$ )

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(also eigenvalues of  $R$ )

$$D^{1/2} v_1 \longleftrightarrow \text{stationary distribution} \longleftrightarrow d$$

$$\Rightarrow v_1 = \frac{\sqrt{d}}{\|\sqrt{d}\|} = \frac{\sqrt{d}}{\sqrt{\langle d, \mathbf{1} \rangle}}$$

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$$\text{w/ } \mu_1 = 1 \geq \mu_2 \geq \dots \geq \mu_n \quad (\text{also eigenvalues of } R)$$

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(also eigenvalues of  $R$ )

$$R^k x = (D^{1/2} Q D^{-1/2})^k x$$

$$= (D^{1/2} Q D^{-1/2})^k x$$

$$= D^{1/2} Q^k D^{-1/2} x$$

$$Q^k =$$

$$\left( \frac{\sqrt{d} \otimes \sqrt{d}}{\langle \mathbf{1}, d \rangle} + \mu_2 (v_2 \otimes v_2) + \dots + \mu_n (v_n \otimes v_n) \right)^k$$

$$= \frac{\sqrt{d} \otimes \sqrt{d}}{\langle \mathbf{1}, d \rangle} + \mu_2^k (v_2 \otimes v_2) + \dots + \mu_n^k (v_n \otimes v_n)$$

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(orthonormal  $v_i$ 's  $\Rightarrow$  cross-terms zero)

$$D^{1/2} \frac{\sqrt{d} \otimes \sqrt{d}}{\langle \mathbf{1}, d \rangle} D^{-1/2} x = \frac{d \otimes \mathbf{1}}{\langle \mathbf{1}, d \rangle} x$$

$$= \frac{\langle \mathbf{1}, x \rangle}{\langle \mathbf{1}, d \rangle} d = \frac{d}{\langle \mathbf{1}, d \rangle}$$

$$D^{1/2} Q^k D^{-1/2} x =$$

$$\frac{d}{\langle \mathbf{1}, d \rangle} + D^{1/2} \underbrace{(\mu_2^k (v_2 \otimes v_2) + \dots + \mu_n^k (v_n \otimes v_n))}_{\text{"S" has eigenvalues } 0, \mu_2^k, \dots, \mu_n^k} D^{-1/2} x$$

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"S" has eigenvalues  $0, \mu_2^k, \dots, \mu_n^k$

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$$\| \mathbb{R}^k x - \frac{d}{\langle \mathbf{1}, d \rangle} \|^2 = \| D^{1/2} S D^{-1/2} x \|^2$$

$$= \langle S D^{-1/2} x, D S D^{-1/2} x \rangle$$

$$\leq \Delta_{\max} \langle S D^{-1/2} x, S D^{-1/2} x \rangle$$

$$= \Delta_{\max} \langle D^{-1/2} x, S^2 D^{-1/2} x \rangle$$

$$\leq \max \{ \mu_2^{2k}, \mu_n^{2k} \} \Delta_{\max} \langle D^{-1/2} x, D^{-1/2} x \rangle$$

$$= \max \{ \mu_2^{2k}, \mu_n^{2k} \} \Delta_{\max} \langle x, D^{-1} x \rangle$$

$$= \max \{ \mu_2^{2k}, \mu_n^{2k} \} \frac{\Delta_{\max}}{\Delta_{\min}} \langle x, x \rangle$$

$$\leq \max \{ \mu_2^{2k}, \mu_n^{2k} \} \frac{\Delta_{\max}}{\Delta_{\min}}$$

$$\max \{\mu_2^{2k}, \mu_n^{2k}\} = \max \{\mu_2, |\mu_n|\}^{2k} \\ = (1-\delta)^{2k}$$

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Lemma let  $A = C^{-1}BC$ .

- $A$  and  $B$  have same eigenvalues
- $C$  maps eigenvectors of  $A$  to eigenvectors of  $B$

Normalized random walk matrix

$$Q = D^{-1/2} R D^{1/2} = D^{-1/2} A D^{1/2}$$

- $D = \text{diag}(d)$  w/  $d = \text{degrees}$
- $A = \text{adjacency matrix}$
- $R = A D^{-1} = \text{random walk matrix}$
- $Q$  is symmetric!
- $Q$  has same eigenvalues as  $R$ !



$$Q = \frac{\sqrt{d} \otimes \sqrt{d}}{\langle \mathbf{1}, d \rangle} + \mu_2 (v_2 \otimes v_2) + \dots + \mu_n (v_n \otimes v_n)$$

w/  $\mu_1 = 1 \geq \mu_2 \geq \dots \geq \mu_n$   
(also eigenvalues of  $R$ )

Theorem. let  $\gamma = 1 - \max\{\mu_2, |\mu_n|\}$

$$\|R^k x - s\| \leq (1 - \gamma)^k \sqrt{\frac{\Delta_{\max}}{\Delta_{\min}}}$$

↑  
stationary distribution

Rank-1 matrix  
 $u \otimes v$  where  $u, v \in \mathbb{R}^n$   
 $\langle x, (u \otimes v) y \rangle = \langle x, u \rangle \langle v, y \rangle$

Lemma:  $L \in \mathbb{R}^{n \times n}$  symmetric,  
 $x \in \mathbb{R}^n$  maximizes  $\langle x, Lx \rangle$  over  $\|x\|=1$   
 $\Rightarrow Lx = \mu x$  for  $\mu = \langle x, Lx \rangle$

Spectral theorem.  $L \in \mathbb{R}^{n \times n}$  symmetric  
 $L = \lambda_1 (v_1 \otimes v_1) + \dots + \lambda_n (v_n \otimes v_n)$   
where  $\lambda_1, \dots, \lambda_n \in \mathbb{R}$   
 $v_1, \dots, v_n$  orthonormal basis

Theorem:  $G=(V, E)$  undirected,  $w: E \rightarrow \mathbb{R}_{>0}$   
then  $R$  has stationary distr.  $s \in \Delta_V$  w/

$$s_v = \frac{\deg(v)}{2W} \quad \deg(v) = \sum_{e \in \mathcal{E}(v)} w(e), \quad W = \sum_{e \in E} w(e)$$

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