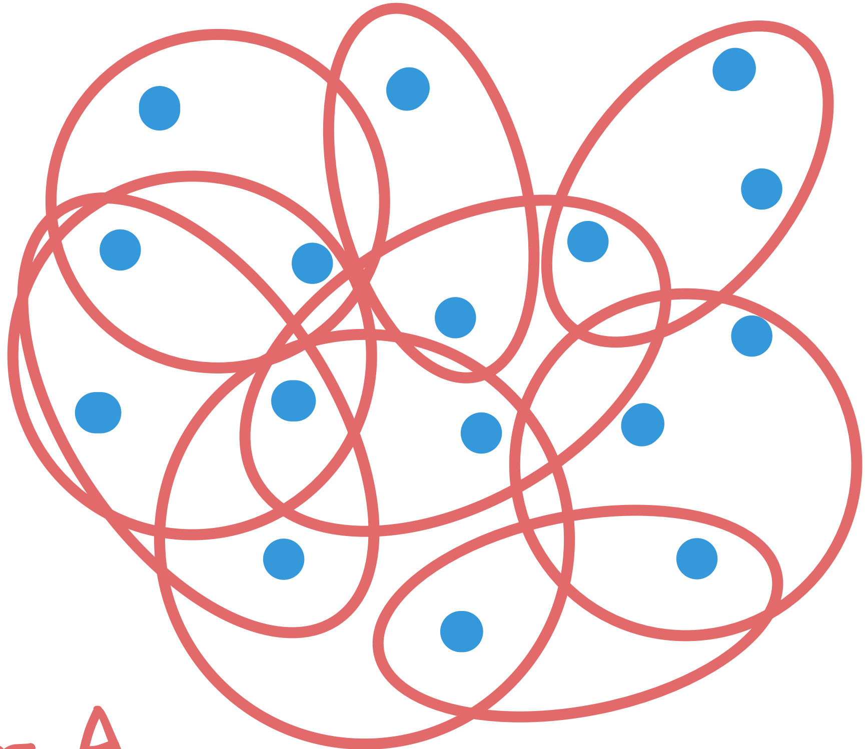
Randomized Greedy

Maximum Coverage



m points $[m] \equiv \{1, \ldots, m\}$

n sets $A_1, \cdots, A_n \subseteq [m]$

cardinality constraint $k \in \mathbb{N}$

Goal: select k sets $A_{e_1}, A_{e_2}, \cdots, A_{e_k}$

maximizing the union $|A_{e_1} \cup A_{e_2} \cup \cdots \cup A_{e_k}|$

NP-Hard via set cover

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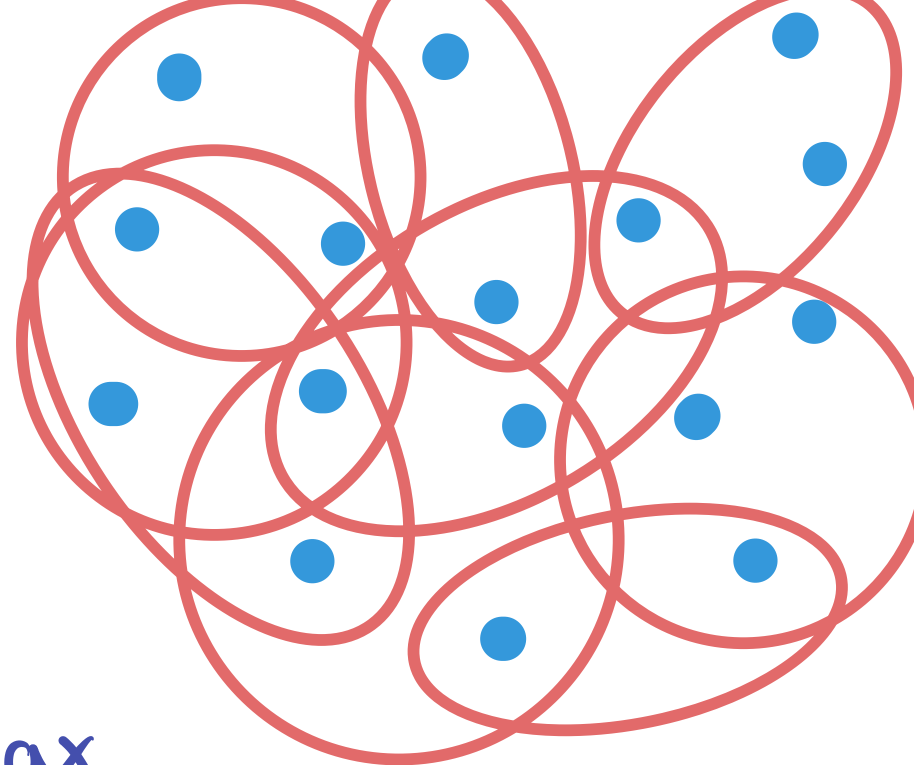


Set Cover

(next time)

minimize # sets to cover all points

NP-Hard (via SAT)

LP + randomized rounding

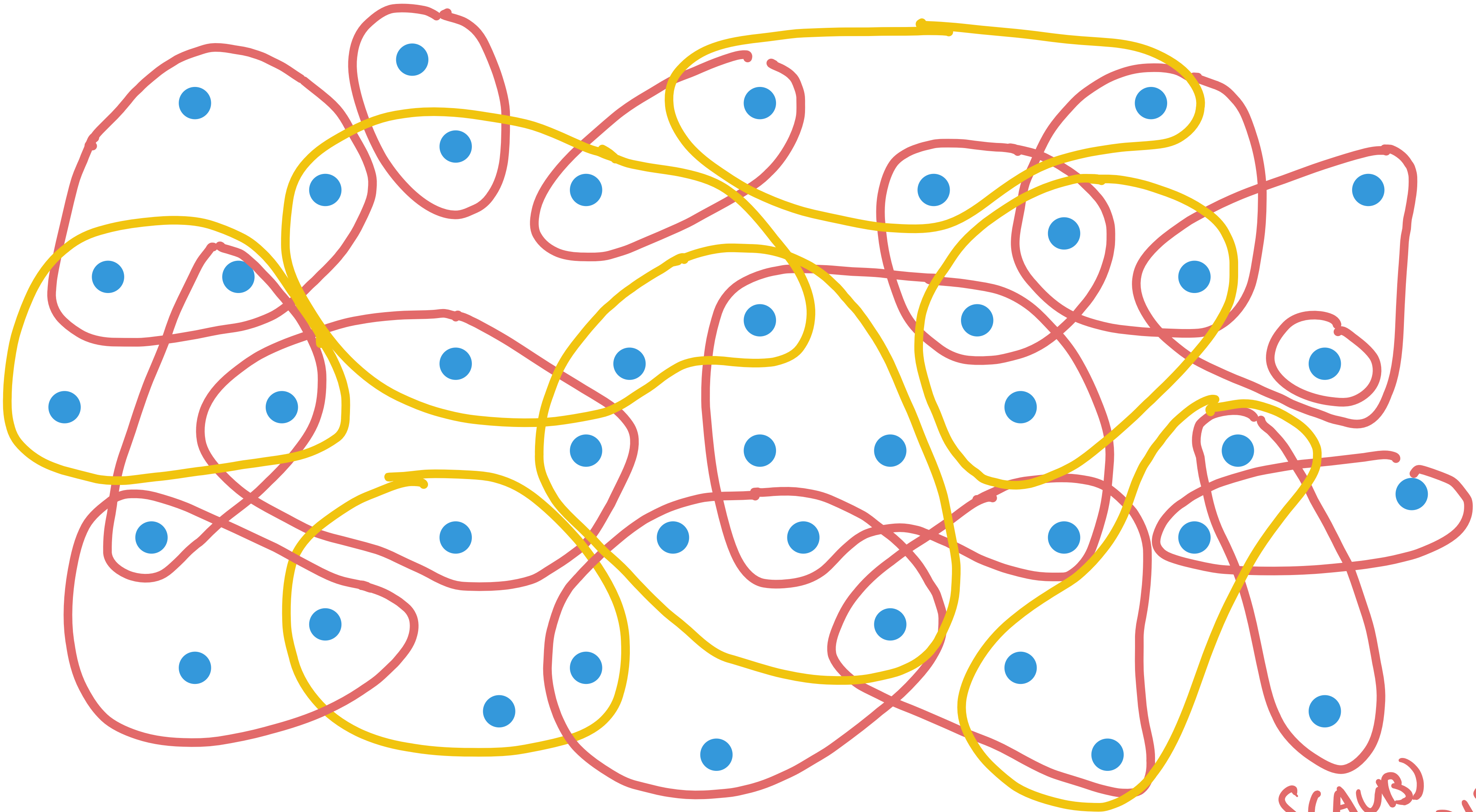
Max Coverage

maximize # points covered w/ $\leq k$ sets

(packing constraint)

(why?) $\Longrightarrow$ NP-Hard

greedy algorithm



$f(A \cup B) \leq f(A) + f(B)$

greedy algorithm repeatedly selects the set covering the most new points

"subadditive"

$f(S) \leq \sum_{e \in S} f(e)$

how good is the greedy algorithm?
(approximation ratio)

Set function $f: 2^N \to \mathbb{R}$ s.t.

- **Normalized:** $f(\emptyset) = 0$
- **Nonnegative:** $f(S) \geq 0 \quad \forall S$
- **Monotone:** $f(S) \leq f(T)$ for $S \subseteq T$
- **Submodular:** for $A, B \subseteq N$,

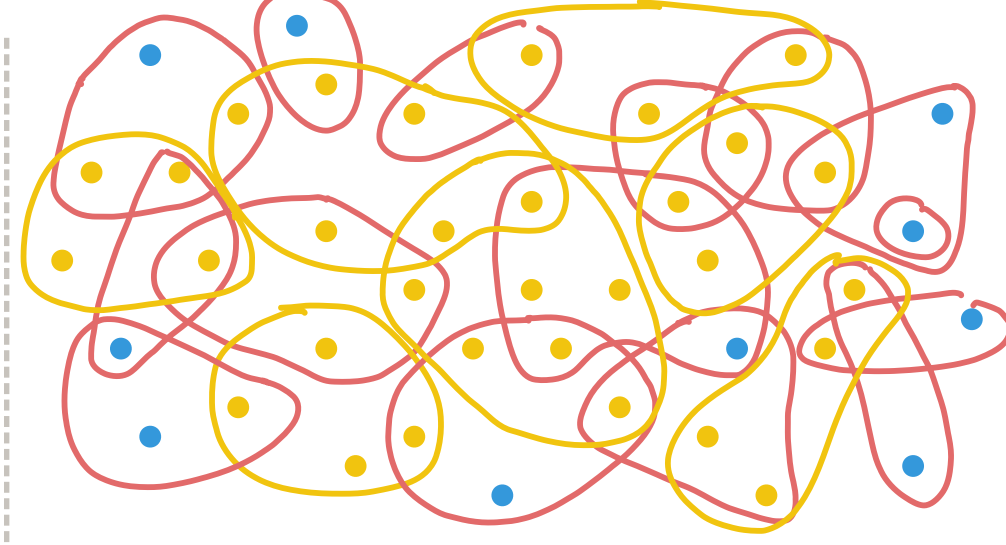
$$f(A) + f(B) \geq f(A \cup B) + f(A \cap B)$$

equivalently:

"decreasing marginal values"

for $S \subseteq T \subseteq N$, $e \in N$,

$$\underbrace{f(S+e) - f(S)}_{\stackrel{\text{def}}{=} f(e|S)} \geq \underbrace{f(T+e) - f(T)}_{\stackrel{\text{def}}{=} f(e|T)}$$

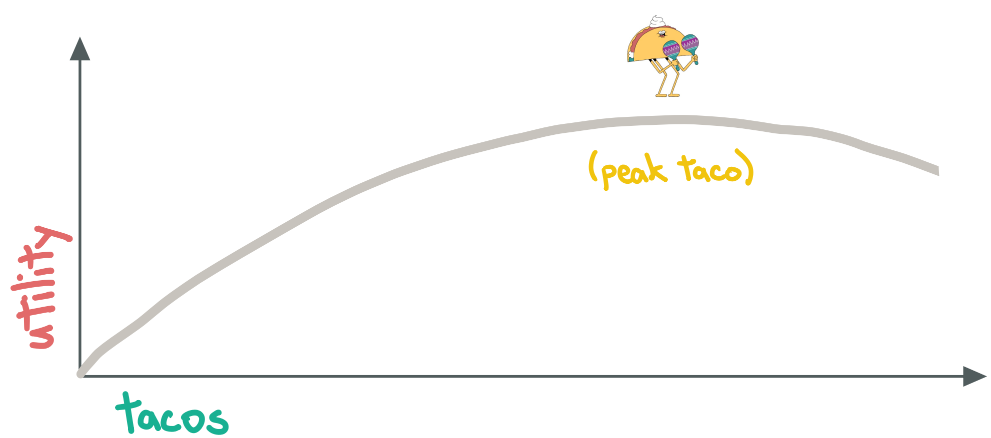


m points $[m] \equiv \{1, \ldots, m\}$

n sets $A_1, \ldots, A_n \subseteq [m]$

$S \subseteq [n]$ interpreted as collection of sets (via indices)

$$f(S) \stackrel{\text{def}}{=} \left(\begin{array}{c} \#\text{ points covered by} \\ \text{sets (w/ index in) } S \end{array} \right)$$



greedy$(f, \mathcal{N}, k)$

1. $S_0 \leftarrow \emptyset$.
2. For $i = 1, \dots, k$:
   A. Let $e_i \in \mathcal{N}$ maximize $f(e_i \mid S_{i-1})$
   B. $S_i \leftarrow S_{i-1} + e$.
3. Return S_k.

$T_j = \{d_1, \dots, d_j\}$

$f(T)$ $\quad |T| \le k$

$T = \{d_1, \dots, d_k\}$

Lemma $f(e_i \mid S_{i-1}) \ge \frac{1}{k}(OPT - f(S_{i-1}))$

$$f(S_{i-1}+e_i) - f(S_{i-1}) \ge \frac{1}{k}\sum_{j=1}^{k} f(d_j \mid S_{i-1})$$

$$\ge \frac{1}{k}\sum_{j=1}^{k} f(\underbrace{d_j \mid S_{i-1} \cup T_{j-1}}_{f(T_j \cup S_{i-1}) - f(T_{j-1} \cup S_{i-1})})$$

$$\ge \frac{1}{k}\left(f(S_{i-1}+T) - f(S_{i-1})\right) \ge \frac{1}{k}\left(f(T) - f(S_{i-1})\right)$$

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"decreasing marginal values"

for $S \subseteq T \subseteq \mathcal{N}$, $e \in \mathcal{N}$,

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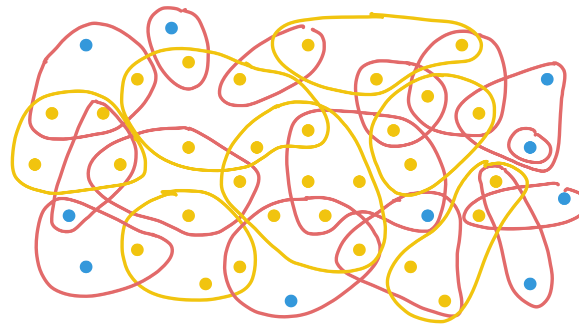

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Lemma $f(e_i \mid S_{i-1}) \geq \frac{1}{k}(OPT - f(S_{i-1}))$

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3. Return S_k.

$+OPT$

$f(S_i) - f(S_{i-1})$

$+OPT$

Lemma $f(e_i \mid S_{i-1}) \geq \frac{1}{k}(OPT - f(S_{i-1}))$

Theorem $f(S_k) \geq (1-\frac{1}{e})OPT$

$$OPT - f(S_i) \leq (1-\tfrac{1}{k})[OPT - f(S_{i-1})]$$

$$OPT - f(S_1) \leq (1-\tfrac{1}{k})[OPT - \cancel{f(\emptyset)}]$$

$$OPT - f(S_2) \leq (1-\tfrac{1}{k})(\quad)$$

$$\leq (1-\tfrac{1}{k})^2 OPT$$

$$\vdots \qquad \vdots$$

$$OPT - f(S_k) \leq (1-\tfrac{1}{k})^k OPT \leq e^{-1} OPT$$

$$f(S_k) \geq (1-e^{-1})OPT$$

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Monotone Submodular f

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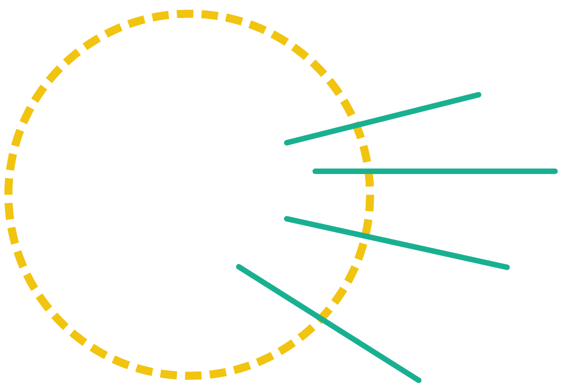
Suppose f was not monotone

e.g. graph cuts: for $S \subseteq V$,

$f(S) = \#$ edges cut by S



does greedy work?

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does greedy work?

e.g. graph cuts

Recall

$\{$

$$f(S^* | S_i) = f(S^* \cup S_i) - f(S_i)$$

$$\overset{\text{monotonicity}}{\geq} f(S^*) - f(S_i)$$

f not monotone: $f(S^* \cup S_i) \ngeq f(S^*)$

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Suppose f was not monotone

Lemma $R \subseteq \mathcal{N}$ random, $P[e \in R] \leq p$ $\forall e$
then
$$E[f(R)] \geq (1-p) f(\emptyset)$$

e.g. if S_i random, $P[e \in S_i] \leq 1/2$ $\forall e$

$$E[f(S^* \cup S_i)] \geq \frac{1}{2} f(S^*)$$

$\underline{g}(S) = f(S \cup S^*)$

$g(\emptyset) = f(S^*)$

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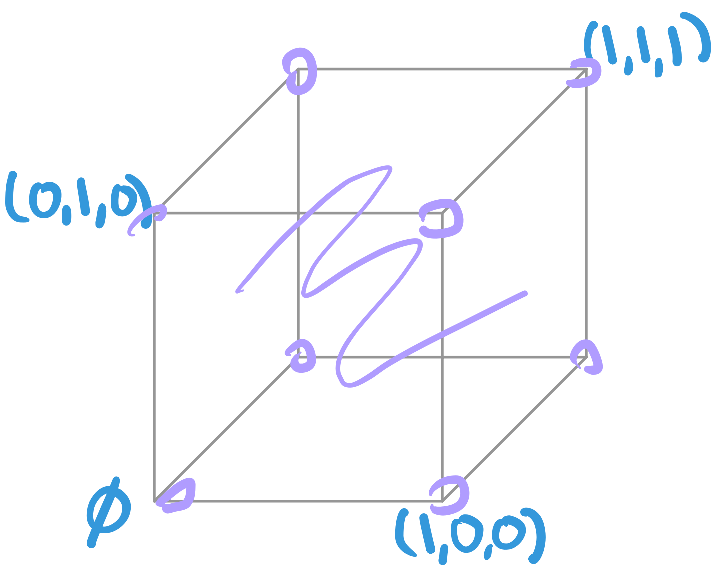
Continuous extensions of f

$F: [0,1]^{\mathcal{N}} \to \mathbb{R}$ extends $f: 2^{\mathcal{N}} \to \mathbb{R}_{\ge 0}$

if $F(\mathbb{1}_S) = f(S)$ $\forall S \subseteq \mathcal{N}$



$\mathbb{1}_S = (0, \ldots, \underset{}{\overset{e \in S}{1}}, \ldots, 0)$

e.g. $F(x) = E[f(S)]$

S independent sample

$\Pr[e \in S] = x_e$

Suppose f was not monotone

Lemma $R \subseteq \mathcal{N}$ random, $P[e \in R] \le p$ $\forall e$ then $E[f(R)] \ge (1-p) f(\emptyset)$

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Convex closure F^-

$$F^-(p) \stackrel{\text{def}}{=} \inf_R \{ E[f(R)] \text{ s.t. } P[e \in R] = p_e \ \forall e \}$$

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Lovász:

$F^-(p) = E[f(S_t)]$ w/ $S_t = \{e : p_e \geq t\}$,

$t \in [0,1]$ unif. random

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$$F^-(p) \overset{\text{def}}{=} \inf\{E[f(R)] \text{ s.t. } P[e\in R]=p_e \ \forall e\}$$

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Lovász: $F^-(p)=E[f(S_t)]$

w/ $S_t=\{e: p_e\geq t\}$, $t\in[0,1]$ unif. random

proof:

(nested)

Observe support(S_t) is a chain

S_t is unique s.t. and margins p

$$R=\arg\inf\{E[f(R)] \text{ s.t. } P[e\in R]=p_e \ \forall e\}$$

claim: R (can be made) is a chain.

suppose not

$p<q$

$A, B \qquad A \cap B \neq \emptyset \qquad A \not\subseteq B \quad B \not\subseteq A$

$P[R = A] = p$

$$\underbrace{f(A)}_{p} + \underbrace{f(B)}_{p} \geq \underbrace{f(A \cup B)}_{p \cdot p} + \underbrace{f(A \cap B)}_{p}$$

choose R max $E[|R|^2]$

R: $\quad P[R = A]$?

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Suppose f was not monotone

Lemma $R \subseteq \mathcal{N}$ random, $P[e\in R] \leq p \ \forall e$

then $E[f(R)] \geq (1-p) f(\emptyset)$

proof $x_e \stackrel{?}{=} P[e\in R] \to S_t$

$$E[f(R)] \geq \underbrace{E[f(S_t)]}$$

$$\geq \geq f(1) + \underbrace{\cdots}_{\geq 0} + (1-p) f(\emptyset)$$

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randomized-greedy(f, k)

1. $S_0 \leftarrow \emptyset$
2. For $i = 1, \ldots, k$:
   A. Let $R_i \subseteq \mathcal{N}$ be the k elements e w/ max $f(e \mid S_{i-1})$.
   B. Sample $e_i \sim R$ uniformly at random.
   C. $S_i \leftarrow S_{i-1} + e_i$.
3. Return S_k.

$$E[f(S_i)] - f(S_{i-1}) = \frac{1}{k} \sum_{e \in R_i} f(e|S_{i-1})$$

$$\geq \frac{1}{k} \sum_{e \in S^*} f(e|S_{i-1}) \geq \frac{1}{k} f(S^* | S_{i-1})$$

$P[e \in S_i]$? $\leq r p^i$

$P[e \notin S_i] \geq (\underbrace{1 - \tfrac{1}{k}}_{p})^i = p^i$

$E[f(S_i)] - E[f(S_{i-1})] \geq \frac{1}{k} E[f(S^* \mid S_{i-1})]$

$\geq \frac{1}{k}\left(p^{i-1} OPT - E[f(S_{i-1})]\right)$

i.e. $E[f(S_i)] \geq \frac{p^{i-1}}{k} OPT + p E[f(S_{i-1})]$

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"decreasing marginal values"

for $S \subseteq T \subseteq \mathcal{N}$, $e \in \mathcal{N}$,

$\underbrace{f(S+e) - f(S)}_{\overset{\text{def}}{=} f(e \mid S)} \geq \underbrace{f(T+e) - f(T)}_{\overset{\text{def}}{=} f(e \mid T)}$

Lemma $R \subseteq \mathcal{N}$ random, $P[e \in R] \leq p \ \forall e$

then $E[f(R)] \geq (1-p) f(\emptyset)$

<u>randomized-greedy(f, k)</u>

1. $S_0 \leftarrow \emptyset$
2. For $i = 1, \ldots, k$:
   - A. Let $R_i \subseteq \mathcal{N}$ be the k elements e w/ max $f(e \mid S_{i-1})$.
   - B. Sample $e_i \sim R$ uniformly at random.
   - C. $S_i \leftarrow S_{i-1} + e_i$.
3. Return S_k.

$$E[f(S_i)] \geq \frac{p^{i-1}}{k}\mathrm{OPT} + pE[f(S_{i-1})]$$

$$p = 1 - \frac{1}{k}$$

$$E[f(S_1)] \geq \frac{1}{k}\mathrm{OPT}$$

$$E[f(S_2)] \geq \frac{p}{k}\mathrm{OPT} + \frac{p}{k}\mathrm{OPT} = \frac{2p}{k}\mathrm{OPT}$$

$$E[f(S_3)] \geq \frac{p^2}{k}\mathrm{OPT} + \frac{2p^2}{k}\mathrm{OPT} = \frac{3p^2}{k}\mathrm{OPT}$$

$$\vdots$$

$$E[f(S_i)] \geq \frac{p^{i-1}}{k}\mathrm{OPT} + \frac{(i-1)p^{i-1}}{k}\mathrm{OPT} = \frac{ip^{i-1}}{k}\mathrm{OPT}$$

$$\vdots$$

$$E[f(S_k)] \geq \frac{kp^{k-1}}{k}\mathrm{OPT}$$

$$= \left(1-\frac{1}{k}\right)^{k-1}\mathrm{OPT} \geq \frac{1}{e}\mathrm{OPT}$$

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$\frac{1}{e}$ approximation

nonnegative

submodular

function

$\frac{1}{e} + .001$

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