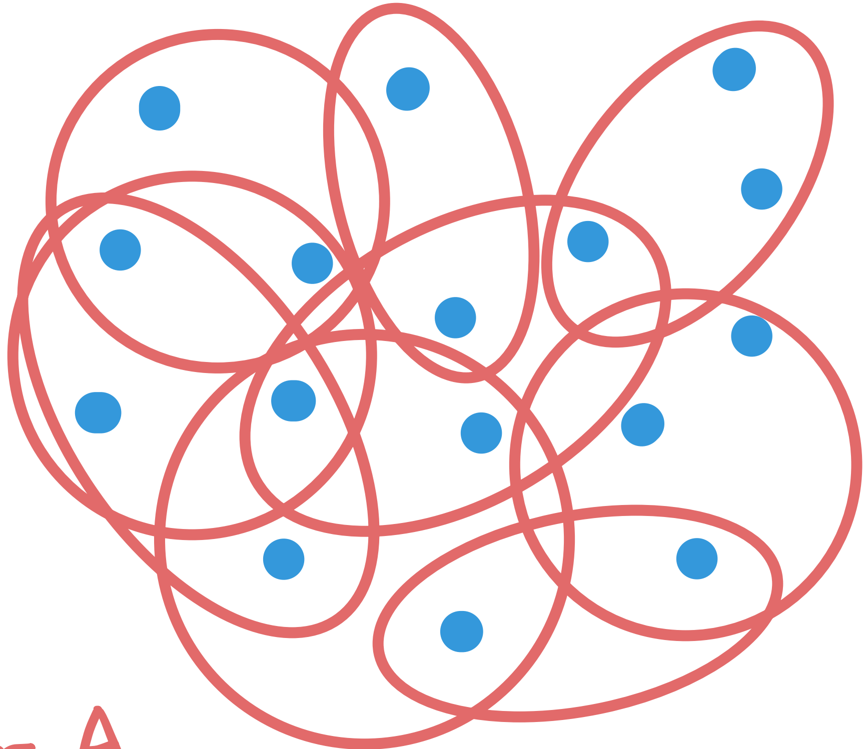
Randomized Greedy

Maximum Coverage



m points $[m] \equiv \{1, \dots, m\}$

n sets $A_1, \dots, A_n \subseteq [m]$

cardinality constraint $k \in \mathbb{N}$

Goal: select k sets $A_{e_1}, A_{e_2}, \dots, A_{e_k}$

maximizing the union $|A_{e_1} \cup A_{e_2} \cup \dots \cup A_{e_k}|$

NP-Hard via set cover

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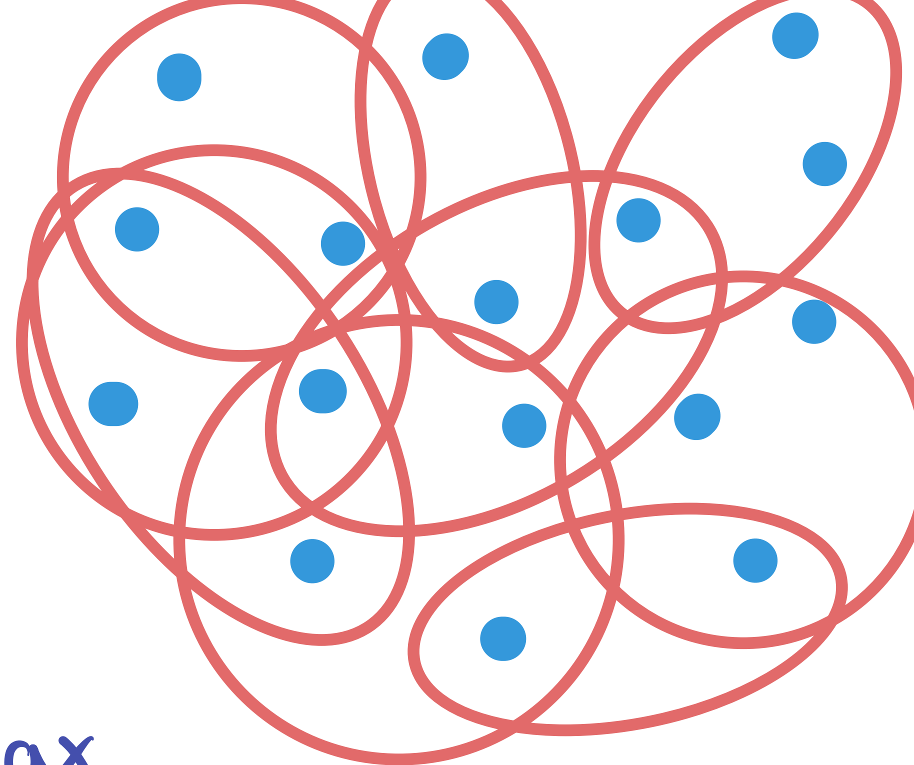


Set Cover
(next time)

minimize # sets to cover all points

NP-Hard (via SAT)

LP + randomized rounding

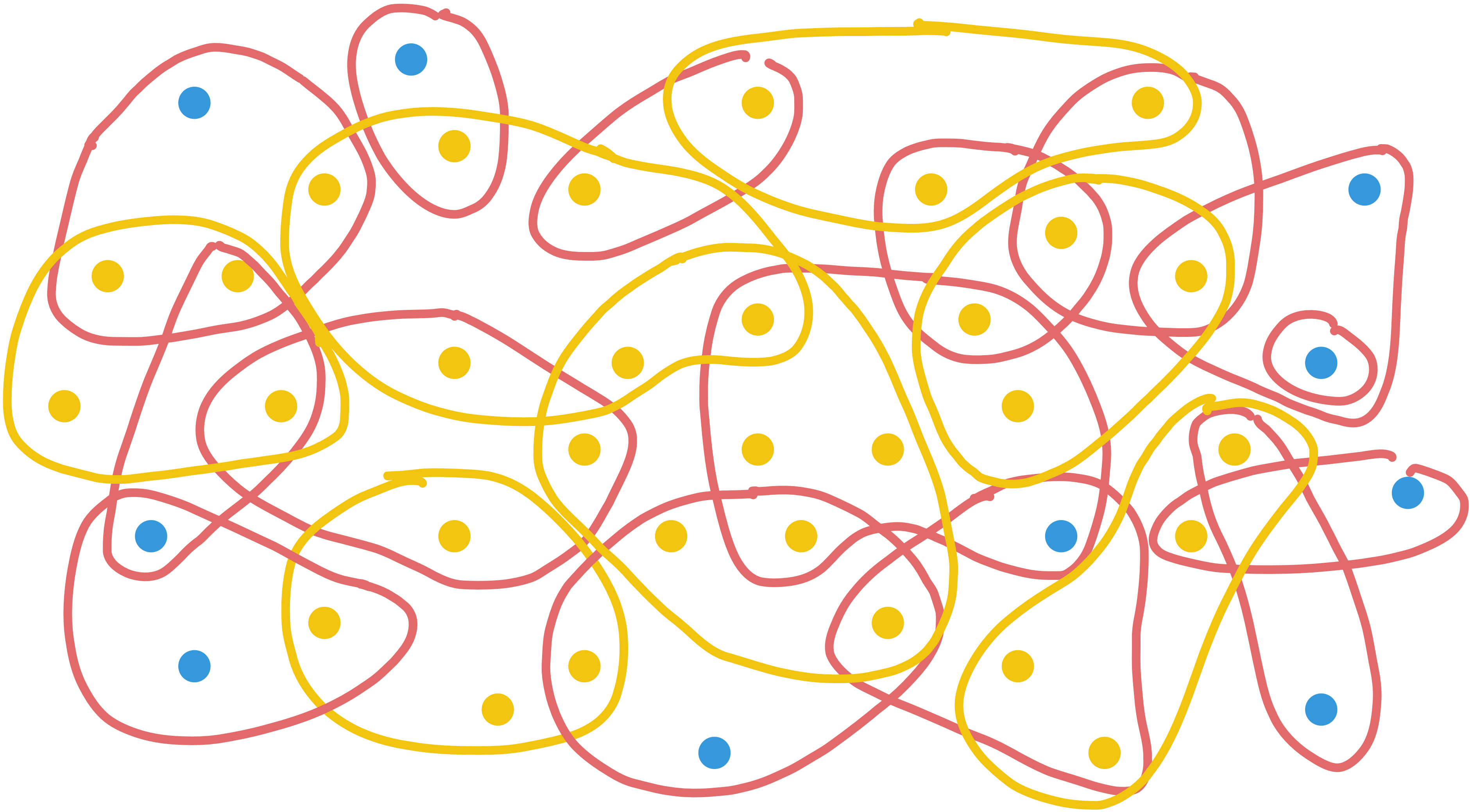
(why?) $\Rightarrow$

Max Coverage

maximize # points covered w/ $\leq k$ sets
(packing constraint)

NP-Hard

greedy algorithm



greedy algorithm repeatedly selects the set covering the most new points

how good (approximation ratio) is the greedy algorithm?

Set function $f: 2^N \to \mathbb{R}_{\geq 0}$ s.t.

- **Normalized:** $f(\emptyset) = 0$
- **Nonnegative:** $f(S) \geq 0 \quad \forall S$
- **Monotone:** $f(S) \leq f(T)$ for $S \subseteq T$
- **Submodular:** for $A, B \subseteq N$,

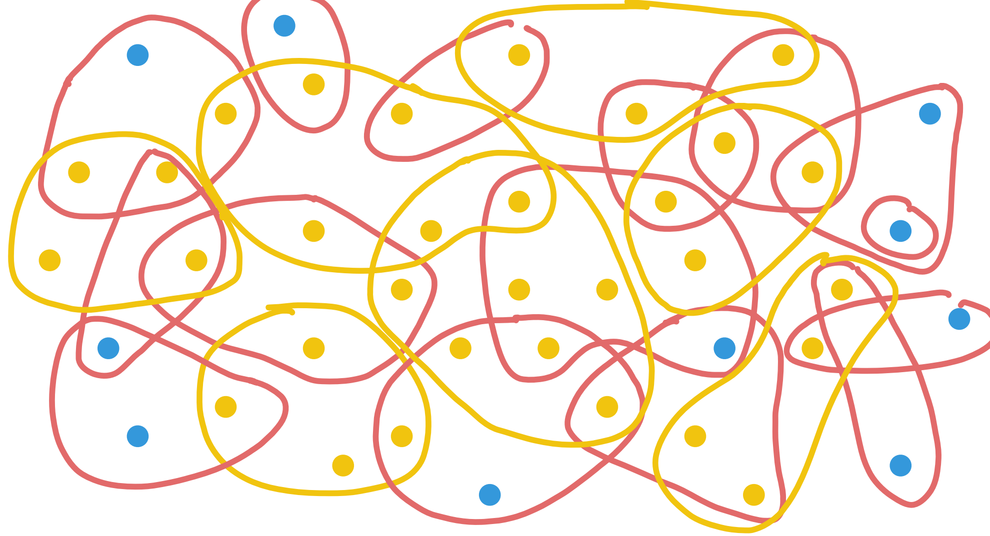
$$f(A) + f(B) \geq f(A \cup B) + f(A \cap B)$$

equivalently:

"decreasing marginal values"

for $S \subseteq T \subseteq N$, $e \in N$,

$$\underbrace{f(S+e) - f(S)}_{\overset{\text{def}}{=} f(e|S)} \geq \underbrace{f(T+e) - f(T)}_{\overset{\text{def}}{=} f(e|T)}$$



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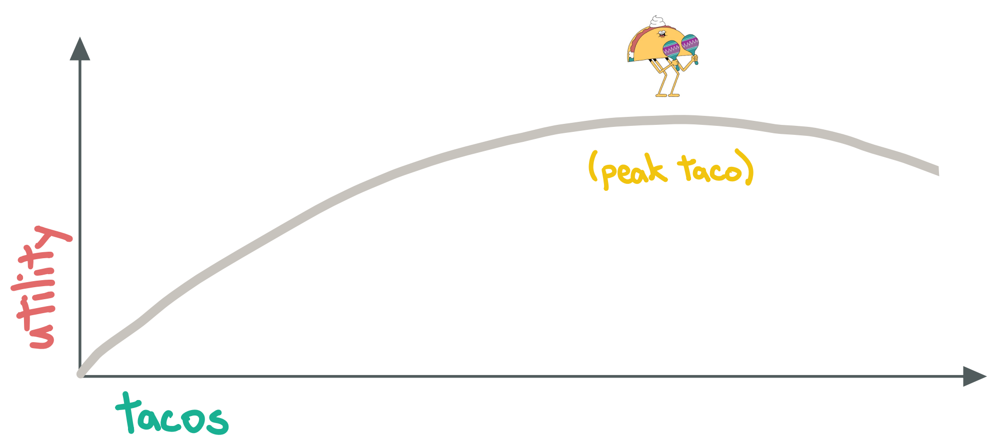
$S \subseteq [n]$ interpreted as collection of sets (via indices)

$$f(S) \overset{\text{def}}{=} \left(\begin{array}{c} \text{\# points covered by} \\ \text{sets (w/ index in) } S \end{array} \right)$$



greedy$(f, \mathcal{N}, k)$

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   A. Let $e_i \in \mathcal{N}$ maximize $f(e_i \mid S_{i-1})$
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Lemma $f(e_i \mid S_{i-1}) \geq \frac{1}{k}(OPT - f(S_{i-1}))$

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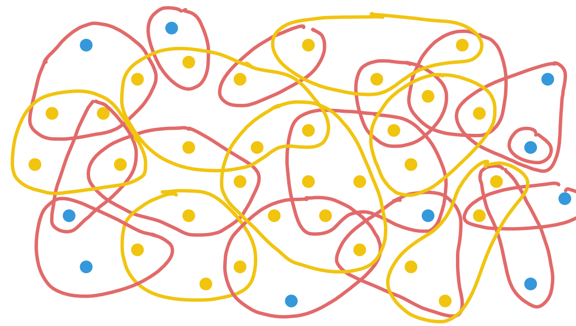

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Suppose f was not monotone

e.g. graph cuts: for $S \subseteq V$,

$f(S) =$ edges cut by S

does greedy work?

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e.g. graph cuts

$$\rightsquigarrow f(S^* \mid S_i) = f(S^* \cup S_i) - f(S_i)$$

$$\overset{\text{monotonicity}}{\geq} f(S^*) - f(S_i)$$

f not monotone: $f(S^* \cup S_i) \not\geq f(S^*)$

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Lemma $R \subseteq \mathcal{N}$ random, $P[e \in R] \leq p \ \forall e$

then

$$E[f(R)] \geq (1-p) f(\emptyset)$$

e.g. if S_i random, $P[e \in S_i] \leq 1/2 \ \forall e$

$$E[f(S^* \cup S_i)] \geq \frac{1}{2} f(S^*)$$

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Continuous extensions of f

$F: [0,1]^{\mathcal{N}} \to \mathbb{R}$ extends $f: 2^{\mathcal{N}} \to \mathbb{R}_{\geq 0}$

if $F(\mathbb{1}_S) = f(S)$ $\forall S \subseteq \mathcal{N}$

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Convex closure F^-

$$F^-(p) \stackrel{\text{def}}{=} \inf \{ E[f(R)] \text{ s.t. } P[e \in R] = p_e \ \forall e \}$$

Lovász:

$$F^-(p) = E[f(S_t)] \quad \text{w/} \quad S_t = \{e : p_e \geq t\},$$
$t \in [0,1]$ unif. random

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w/ $S_t=\{e : p_e \geq t\}$, $t\in[0,1]$ unif. random

proof:

(nested)

Observe support(S_t) is a chain

S_t is unique s.t. and margins p

$$R = \arg\inf\{E[f(R)] \text{ s.t. } P[e\in R]=p_e \ \forall e\}$$

claim: R (can be made) is a chain.

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$$E[f(S_i)] - f(S_{i-1}) = \frac{1}{k} \sum_{e \in R_i} f(e \mid S_{i-1})$$

$$\ge \frac{1}{k} \sum_{e \in S^*} f(e \mid S_{i-1}) \ge \frac{1}{k} f(S^* \mid S_{i-1})$$

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$P[e \in S_i]$?

$$P[e \notin S_i] \geq \underbrace{\left(1-\frac{1}{k}\right)}^{p}{}^{i} = p^i$$

$$E[f(S_i)] - E[f(S_{i-1})] \geq \frac{1}{k} f(S^* \mid S_{i-1})$$

$$\geq \frac{1}{k}\left(p^{i-1} OPT - E[f(S_{i-1})]\right)$$

i.e. $E[f(S_i)] \geq \frac{p^{i-1}}{k} OPT + p\, E[f(S_{i-1})]$

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$$E[f(S_i)] \geq \frac{p^{i-1}}{k} OPT + p\, E[f(S_{i-1})]$$

$$p = 1 - \frac{1}{k}$$

$$E[f(S_1)] \geq \frac{1}{k} OPT$$

$$E[f(S_2)] \geq \frac{p}{k} OPT + \frac{p}{k} OPT = \frac{2p}{k} OPT$$

$$E[f(S_3)] \geq \frac{p^2}{k} OPT + \frac{2p^2}{k} OPT = \frac{3p^2}{k} OPT$$

$$\vdots$$

$$E[f(S_i)] \geq \frac{p^{i-1}}{k} OPT + \frac{(i-1)p^{i-1}}{k} OPT = \frac{ip^{i-1}}{k} OPT$$

$$\vdots$$

$$E[f(S_k)] \geq \frac{kp^{k-1}}{k} OPT$$

$$= \left(1 - \frac{1}{k}\right)^{k-1} OPT \geq \frac{1}{e} OPT$$

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   A. Let $R_i \subseteq \mathcal{N}$ be the k elements e w/ max $f(e \mid S_{i-1})$.
   B. Sample $e_i \sim R$ uniformly at random.
   C. $S_i \leftarrow S_{i-1} + e_i$.
3. Return S_k.

Lemma $R \subseteq \mathcal{N}$ random, $P[e \in R] \leq p \ \forall e$
then $E[f(R)] \geq (1-p) f(\emptyset)$

Set function $f: 2^{\mathcal{N}} \to \mathbb{R}_{\geq 0}$ s.t.

- Normalized: $f(\emptyset) = 0$
- Nonnegative: $f(S) \geq 0 \quad \forall S$
- Monotone: $f(S) \leq f(T)$ for $S \subseteq T$
- Submodular: for $A, B \subseteq \mathcal{N}$,

$f(A) + f(B) \geq f(A \cup B) + f(A \cap B)$

"decreasing marginal values"

for $S \subseteq T \subseteq \mathcal{N}$, $e \in \mathcal{N}$,

$$\underbrace{f(S+e) - f(S)}_{\overset{\text{def}}{=} f(e|S)} \geq \underbrace{f(T+e) - f(T)}_{\overset{\text{def}}{=} f(e|T)}$$

<u>`randomized-greedy`(f, k)</u>

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