

# Randomized Rounding

# Lecture 1

## Probability

Events, probabilities,  
conditional probabilities,  
union bound

Random variables,  
expected value,  
linearity of expectation  
Markov's inequality

## Algorithms

### Approx. 3-SAT,

Big idea: we only want the average

$$\begin{array}{l} \text{Linearity of Expectation} \\ E[X+Y] = E[X] + E[Y] \quad (\text{stochastic variable}) \\ \text{"average sum = sum of averages"} \end{array}$$

let  $Y_i = \begin{cases} 1 & \text{if } i\text{th clause satisfied} \\ 0 & \text{otherwise} \end{cases}$  ( $i=1, \dots, m$ )

$$E[Y_i] = 7/8 \quad (1 \text{ clause case})$$

$$E[\text{\# clauses satisfied}] = E\left[\sum_{i=1}^m Y_i\right] = \sum_{i=1}^m E[Y_i] = 7/8 m$$

let  $Y_i = \begin{cases} 1 & \text{if } i\text{th clause satisfied} \\ 0 & \text{otherwise} \end{cases}$

$$Z = \sum_{i=1}^m Y_i = \text{\# clauses satisfied}$$

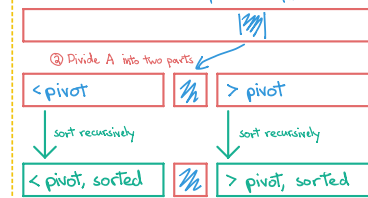
$$E[Z] = \frac{1}{2} E[Z | x_1 = \text{true}] + \frac{1}{2} E[Z | x_1 = \text{false}]$$

$\Rightarrow$  either  $E[Z | x_1 = \text{true}]$  or  $E[Z | x_1 = \text{false}]$  is  $\geq E[Z]$  !

### QuickSort,

Random divide and conquer

① pick random 'pivot'



Expected running time of QuickSort

For each  $i, j \in [n]$  w/  $i < j$ ,  
 $X_{ij} = \begin{cases} 1 & \text{if } i\text{th smallest element is compared to } j\text{th smallest} \\ 0 & \text{otherwise} \end{cases}$

$\sum_{i,j} X_{ij}$  = total # comparisons made by the algorithm  
 $\approx$  total running time

$E[\sum_{i,j} X_{ij}]$  = average running time

$$E[\sum_{i,j} X_{ij}] = \sum_{i,j} E[X_{ij}]$$

now analyze  $E[X_{ij}]$  in isolation

$$E[X_{ij}] = \Pr[X_{ij} = 1] = \Pr[i\text{th element is compared to } j\text{th}]$$

$i$ th and  $j$ th elem selected before any element in between happens w/ probability  $\frac{2}{j-i+1}$

$$E[X_{ij}] = \frac{2}{j-i+1}$$

$$E[\sum_{i,j} X_{ij}] = \sum_{i,j} \frac{2}{j-i+1}$$

$$\begin{aligned} &= \sum_{i=1}^n \sum_{j=i+1}^n \frac{2}{j-i+1} \\ &\leq 2n \left( \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n} \right) \\ &\leq O(n \log n) \quad \text{with harmonic series} \end{aligned}$$

### Quickselect (HW)

# Lecture 2

## Probability

### Hash Functions,

Universal Hash Functions

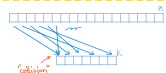
$h: [n] \rightarrow k$  s.t. for all pairs  $i, j \in [n]$ ,  
 $\Pr[h(i) = h(j)] \leq \frac{1}{k}$  (much weaker than ideal hash)

e.g.  $h(x) = (ax + b) \bmod p \bmod k$

where •  $p$  prime,  $> k$

•  $a$  in  $\{1, \dots, p-1\}$  unif. random

•  $b$  in  $\{0, \dots, p-1\}$  unif. random



## Markov's Inequality

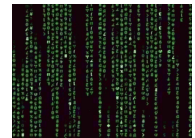
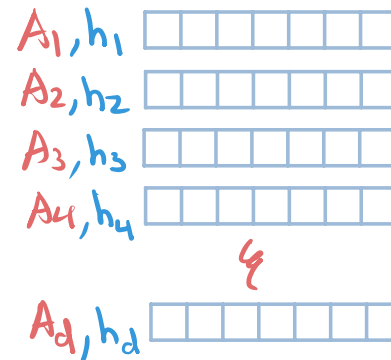
Markov's Inequality (special case)

Let  $X \geq 0$  be nonnegative RV

$$\Pr[X \geq 2E[X]] \leq \frac{1}{2}$$

## Algorithms

### Frequency Estimation, Heavy Hitters (in streams)



estimate  $f_i$  w/  
 $\min_{j=1, \dots, d} A_j[h_j(i)]$

# Lecture 3 (last Tuesday)

## Probability

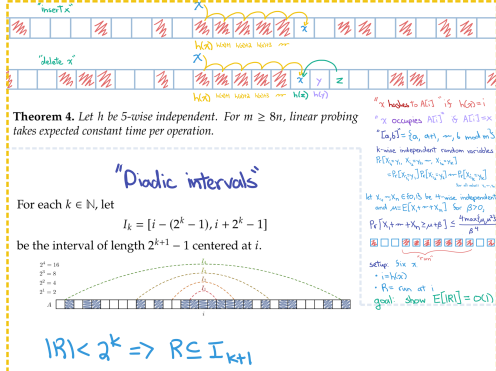
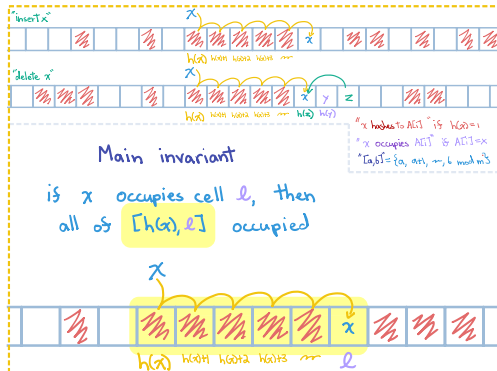
k-wise independence,  
higher order moments  
(generalizing universal hash  
functions and Markov's inequality)

## Algorithms

Hash tables w/  
chaining +  
linear probing

k-wise independent hash functions  
 $h: [U] \rightarrow [m]$   
s.t. for any  $k$  keys  $x_1, x_2, \dots, x_k \in [U]$ ,  
 $k$  values  $v_1, v_2, \dots, v_k \in [m]$ ,  
 $P[h(x_1)=v_1, h(x_2)=v_2, \dots, h(x_k)=v_k] = \frac{1}{m^k}$   
eg.  $h(x) = a_0 + a_1x + a_2x^2 + \dots + a_{k-1}x^{k-1} \pmod p$   
where  $p \geq U$  prime,  
 $a_0, a_1, \dots, a_{k-1} \in [p]$  indep, uniformly random

Important lemma  
k-wise independent random variables  
 $P[X_1=y_1, X_2=y_2, \dots, X_k=y_k]$   
 $= P[X_1=y_1]P[X_2=y_2] \dots P[X_k=y_k]$   
for all values  $y_1, \dots, y_k$   
let  $X_1, \dots, X_n \in \{0,1\}$  be 4-wise independent  
and  $\mu = E[X_1 + \dots + X_n]$ . then for  $\beta > 0$ ,  
 $P[X_1 + \dots + X_n \geq \mu + \beta] \leq \frac{4 \max\{\mu, \mu^3\}}{\beta^4}$

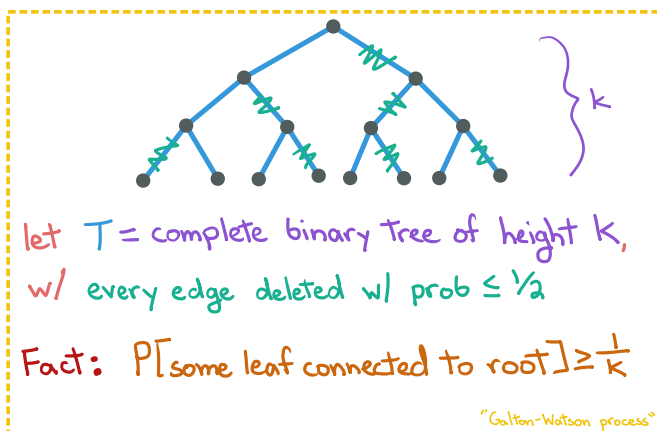




# Lecture 4 (last Thursday)

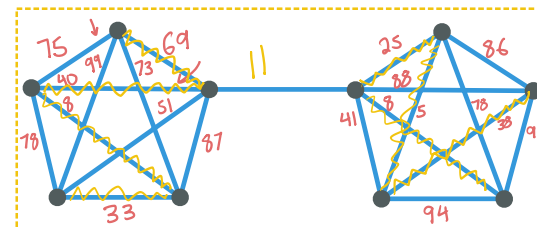
## Probability

### Galton-Watson processes



## Algorithms

### Random Contractions for min cut



#### Randomized contractions

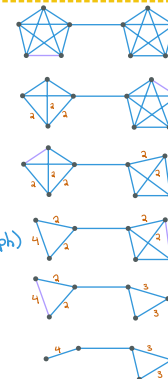
Input:  $G=(V,E)$ , weights  $c \in \mathbb{R}_{\geq 0}^E$

① while  $|E| > 1$

② sample  $e \in E$  in proportion to  $c$

③ "contract"  $e$

④ return set of edges (in original graph) contracted to remaining edge



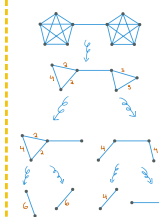
#### Amplification by branching

Instead of restarting at the beginning, restart partway through

#### High-level algorithm

① randomly contract until prob. of ruining the min-cut is  $\sim 1/2$

② Branch:  
make two copies of current graph  
recurse on both copies



# Lecture 5

## Probability

### Multiplicative Chernoff Bound

Multiplicative Chernoff bound

let  $X_1, \dots, X_n \in [0, 1]$  be independent

$$\mu = E[X_1 + \dots + X_n]$$

$$\epsilon \in (0, 1)$$

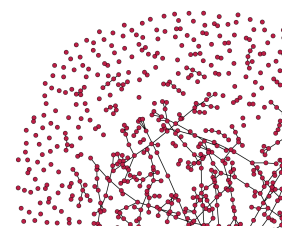
- $P[X_1 + \dots + X_n \geq (1 + \epsilon)\mu] \leq e^{-\epsilon^2 \mu / 3}$
- $P[X_1 + \dots + X_n \leq (1 - \epsilon)\mu] \leq e^{-\epsilon^2 \mu / 2}$

## Algorithms (/ results)

### Random graphs & the giant component

Erdős-Rényi graphs  $G(n, p)$

- $n$  vertices
- each edge sampled independently w/ probability  $p$



**Theorem 5.3.** Consider a random graph  $G \sim \mathcal{G}(n, p)$  for  $p = c/n$ , where  $c$  is a constant.

- If  $c > 1$ , then with high probability, there is exactly one connected component of  $G$  with  $\Omega(n)$  vertices, and all other components have size  $\leq O(\log n)$ .
- For  $c < 1$ , then with high probability, all connected components of  $G$  has size  $< O(\log n)$ .

# Lecture 6 (today)

Probability (& techniques)

Approximations via  
linear programming  
& randomized rounding

Algorithms

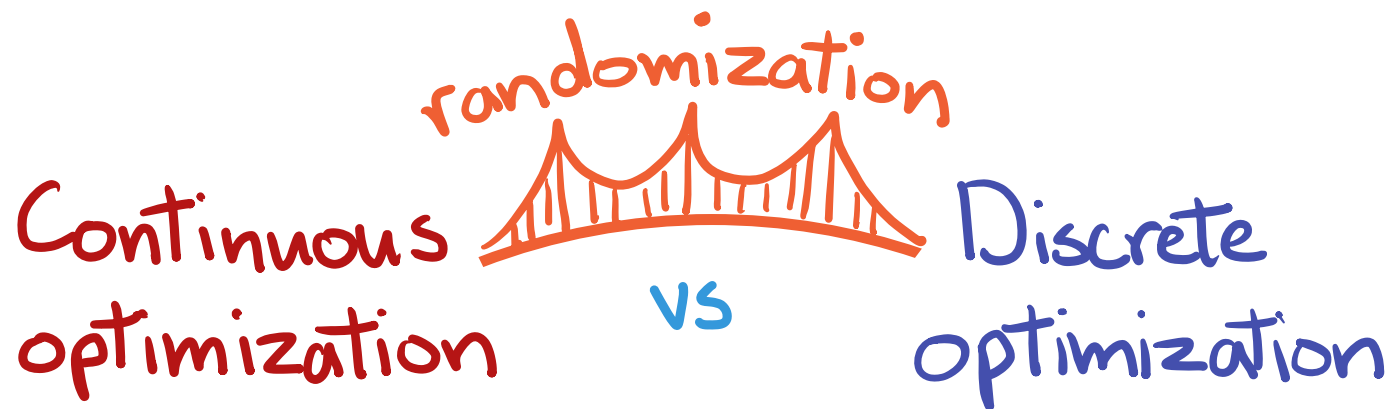
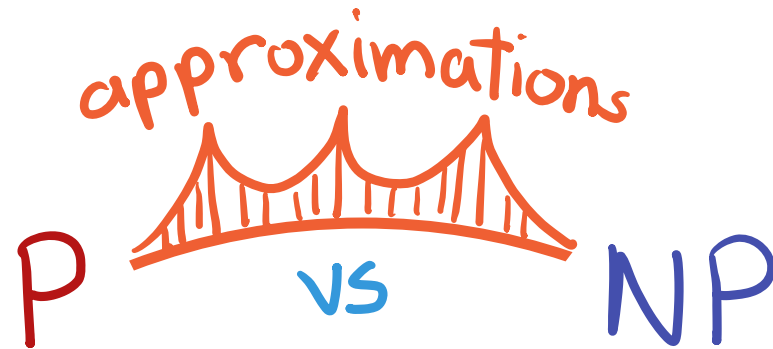
Max Sat

Set Cover

Covering  
Integer Programs

# Randomized Rounding

Story of



# Integer Programs (IPs)

min/max  $\sum_{j=1}^n c_j x_j$

linear objective

over  $x \in \mathbb{Z}^n$

integral variables

s.t.  $A_1 x \leq b_1$   
 $A_2 x = b_2$   
 $A_3 x \geq b_3$

linear  
constraints

for fixed vectors  $c, b_1, b_2, b_3$  and matrices  $A_1, A_2, A_3$

NP-Hard

# Linear programs (LPs)

similar to IPs, except no integrality constraints

$$\min/\max \quad \sum_{j=1}^n c_j x_j$$

$$\text{Over } x \in \mathbb{R}^n$$

$$\text{s.t. } A_1 x \leq b_1$$

$$A_2 x = b_2$$

$$A_3 x \geq b_3$$

linear objective

~~linear~~  
~~integral~~ variables

linear  
constraints

for fixed vectors  $c, b_1, b_2, b_3$  and matrices  $A_1, A_2, A_3$

"linear program"

$$\max \underbrace{\sum_{(s,w) \in E} f(s,w) - \sum_{(u,s) \in E} f(u,s)}_{\text{size of flow}} \quad \text{over } f \in \mathbb{R}^E$$

"capacity constraints"

$$\text{s.t. } 0 \leq f(e) \leq c(e) \quad \forall e \in E$$

"conservation of flow"

$$\sum_{(u,v) \in E} f(u,v) = \sum_{(v,w) \in E} f(v,w) \quad \forall v \notin \{s,t\}$$

"s-t flow"  $f: E \rightarrow \mathbb{R}_{\geq 0}$  s.t.

(a) "capacity constraints"  
 $0 \leq f(e) \leq c(e) \quad \forall e \in E$

(b) "conservation of flow"  
 $\sum_{(u,v) \in E} f(u,v) = \sum_{(v,w) \in E} f(v,w) \quad \forall v \notin \{s,t\}$



"linear program" because objective, constraints, all linear functions of  $f$

Linear Programs are in P (!!)

↑  
(weakly polynomial)

1947 - Simplex algorithm  
- Dantzig

(not polytime but practical)

1979 - Ellipsoid algorithm  
- Khachiyan

polytime

1984 - Interior point algorithm  
- Karmarkar



what's the difference?

Discrete  
optimization

vs

Continuous  
optimization

Integer  
Programs

vs

Linear  
Programs



NP-Hard



polynomial time!

# LP-Based Approximations

① model as IP

$$\begin{array}{ll}\max & \sum_{j=1}^n c_j x_j \\ \text{Over} & x \in \mathbb{Z}^n \\ \text{s.t.} & A_1 x \leq b_1\end{array}$$

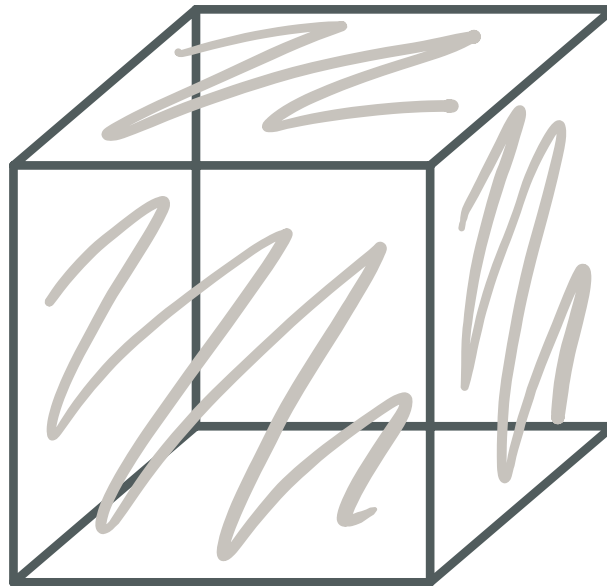


$$\begin{array}{ll}\max & \sum_{j=1}^n c_j x_j \\ \text{Over} & x \in \mathbb{R}^n \\ \text{s.t.} & A_1 x \leq b_1\end{array}$$

② relax to LP



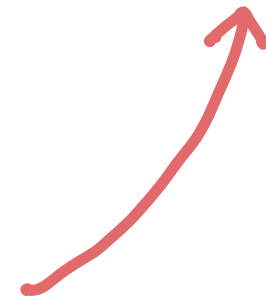
LP solver



approximate  
integral solution  
 $z \in \mathbb{Z}^n$

rounding

optimal  
LP solution  
 $x \in \mathbb{R}^n$



today

Probability (& techniques)

Approximations via  
linear programming  
& randomized rounding

Algorithms

Max Sat

Set Cover

Covering  
Integer Programs

## 3-SAT

"conjunctive normal form"

CNF formula

$$\varphi(x_1, \dots, x_n) = (x_1 \vee \bar{x}_2 \vee x_3) \wedge (x_4 \vee \dots) \wedge \dots$$

$m$  clauses,  $n$  variables

3 (distinct) variables per clause

## The silliest APX algorithm

1. for  $i = 1, \dots, n$

a. randomly set  $x_i = \begin{cases} \text{TRUE} & \text{w/ prob .5} \\ \text{FALSE} & \text{w/ prob .5} \end{cases}$

We will:

① prove silly algo is  $\frac{7}{8}$ -APX

② derandomize silly algo  $\Rightarrow$  deterministic  $\frac{7}{8}$ -APX

Even more surprising:

"probabilistically checkable proofs"

PCP Theorem

$(\frac{7}{8} + \epsilon)$ -APX is NP-Hard  $\forall \epsilon > 0$

## Approximating 3SAT: "Max 3-SAT"

$$\varphi(x_1, \dots, x_n) = (x_1 \vee \bar{x}_2 \vee x_3) \wedge (x_4 \vee \dots) \wedge \dots$$

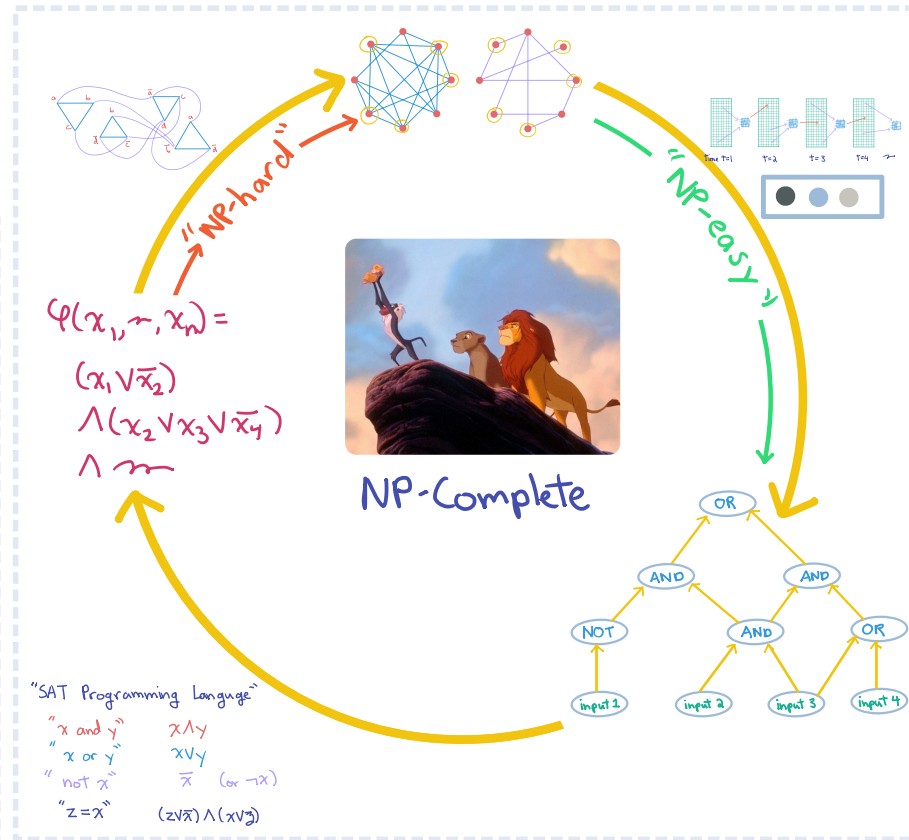
Goal: satisfy Max # clauses

Interesting even if  $\varphi$  is unsatisfiable

"d-Approximation algorithm"  $d \in (0, 1)$

Algorithm satisfies  $d \cdot \text{OPT}$  clauses

Optimum (max) value



2 disjoint clauses

① prove silly algo is  $7/8$ -APX

1. for  $i=1, \dots, n$  The silliest APX algorithm  
 a. randomly set  $x_i = \begin{cases} \text{TRUE} & \text{w/ prob. } .5 \\ \text{FALSE} & \text{w/ prob. } .5 \end{cases}$

$$\mathcal{S}(x) = (\underbrace{x_1 \vee x_2 \vee \bar{x}_3}_{C_1}) \wedge (\underbrace{x_4 \vee \bar{x}_5 \vee x_6}_{C_2})$$

$E_1$  = event that  $C_1$  satisfied

$E_2$  = ~~~~~  $C_2$  ~~~~~

$$\Pr[E_1, E_2] = \frac{1}{8} \cdot \frac{1}{8}$$

$$= \Pr[E_1] \cdot \Pr[E_2]$$

2 clauses w/  
1 common variable

① prove silly algo is  $7/8$ -APX

1. for  $i=1, \dots, n$  The silliest APX algorithm  
 a. randomly set  $x_i = \begin{cases} \text{TRUE} & \text{w/ prob. } .5 \\ \text{FALSE} & \text{w/ prob. } .5 \end{cases}$

$$\mathcal{S}(x) = (\underbrace{x_1 \vee \bar{x}_2 \vee x_3}_{C_1}) \wedge (\underbrace{x_3 \vee x_4 \vee \bar{x}_5}_{C_2})$$

$E_1$  = event that  $C_1$  satisfied

$E_2$  = ~~~~~  $C_2$  ~~~~~

$$\Pr[E_1, E_2] =$$

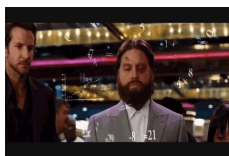
$$\underbrace{\Pr[E_1, E_2 | x_3 = T] \cdot \frac{1}{2}}_1 + \underbrace{\Pr[E_1, E_2 | x_3 = F] \cdot \frac{1}{2}}_{(\frac{3}{4})^2}$$

$m$  clauses w/  
common variables

① prove silly algo is  $7/8$ -APX

1. for  $i=1, \dots, n$  The silliest APX algorithm  
 a. randomly set  $x_i = \begin{cases} \text{TRUE} & \text{w/ prob. } .5 \\ \text{FALSE} & \text{w/ prob. } .5 \end{cases}$

$$\mathcal{S}(x) = (\underbrace{x_1 \vee \bar{x}_2 \vee x_3}_{C_1}) \wedge (\underbrace{x_3 \vee x_4 \vee \bar{x}_5}_{C_2}) \wedge (\underbrace{x_2 \vee \bar{x}_4 \vee x_6}_{C_3}) \wedge \dots$$



$E_1$  = event that  $C_1$  satisfied

$E_2$  = ~~~~~  $C_2$  ~~~~~

$E_3$  = ~~~~~  $C_3$  ~~~~~

$\vdots$

general question

how to analyze a randomized mechanism  
w/ a combinatorial explosion of possibilities?

Big idea: we only want the average

Linearity of Expectation

$$E[X+Y] = E[X] + E[Y] \quad (\text{ridiculously useful})$$

"average sum = sum of averages"

$$\text{let } Y_i = \begin{cases} 1 & \text{if } i\text{th clause satisfied} \\ 0 & \text{otherwise} \end{cases}$$

( $i=1, \dots, m$ )

$$E[Y_i] = 7/8 \quad (1 \text{ clause case})$$

$$E[\text{\# clauses satisfied}] = E\left[\sum_{i=1}^m Y_i\right] = \sum_{i=1}^m E[Y_i] = 7/8 m$$

(actually easy!)

# Max-SAT

"conjunctive normal form"

CNF formula

$$\varphi(x_1, \dots, x_n) = (x_1 \vee \bar{x}_2 \vee x_3) \wedge (x_4 \vee \dots) \wedge \dots$$

m clauses, n variables

any # of variables per clause

goal: satisfy max # clauses

# Approximation ratios for k-SAT vs max-SAT

k-sat

max-SAT

k oblivious APX

$$1 \quad 1/2$$

$$2 \quad 3/4$$

$$3 \quad 7/8$$

$$4 \quad 15/16$$

$$5 \quad 31/32$$

$\vdots$

$$k \quad 1 - 1/2^k$$

$$1/2$$

better?

# LP-Based Approximations

① model as IP

$$\begin{array}{ll}\max & \sum_{j=1}^n c_j x_j \\ \text{Over} & x \in \mathbb{Z}^n \\ \text{s.t.} & A_1 x \leq b_1\end{array}$$

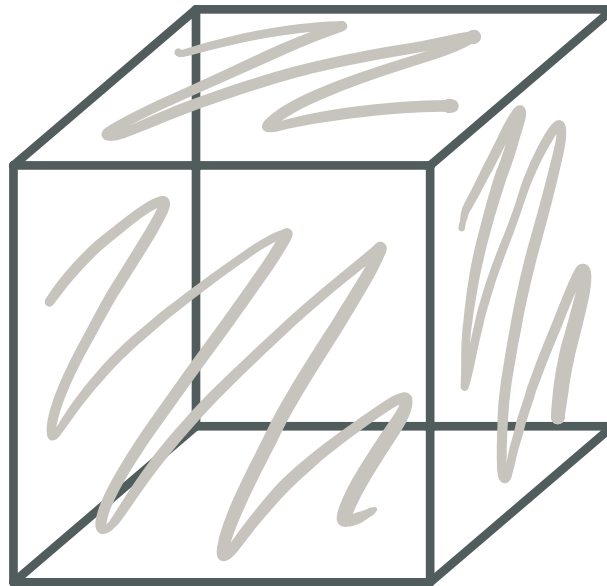


$$\begin{array}{ll}\max & \sum_{j=1}^n c_j x_j \\ \text{Over} & x \in \mathbb{R}^n \\ \text{s.t.} & A_1 x \leq b_1\end{array}$$

② relax to LP



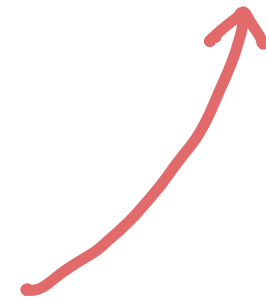
LP solver



approximate  
integral solution  
 $z \in \mathbb{Z}^n$

rounding

optimal  
LP solution  
 $x \in \mathbb{R}^n$





# IP for max-SAT

$$\max \sum_{i=1}^m z_i$$

$$\text{over } y \in \{0,1\}^n, z \in \{0,1\}^m$$

$$\text{s.t. } \sum_{j: x_j \in C_i} y_j + \sum_{j: \bar{x}_j \in C_i} (1 - y_j) \geq z_i$$

for each clause  $C_i$

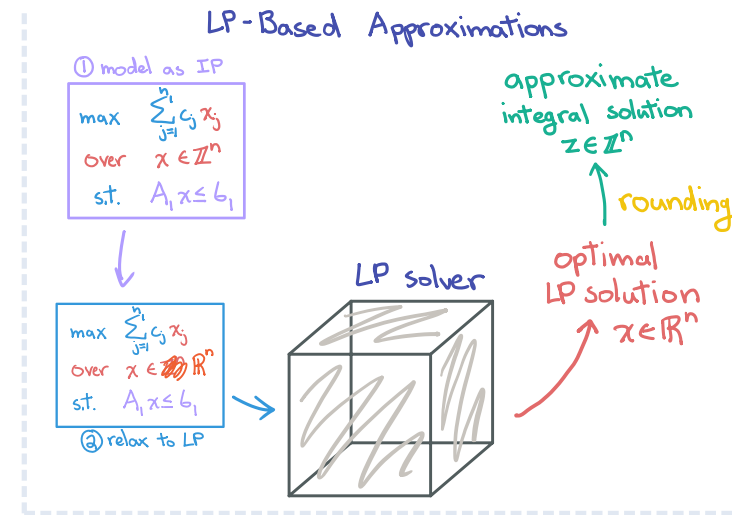
$$z_i = \begin{cases} 1 & \text{if } i\text{th clause satisfied} \\ 0 & \text{if not} \end{cases}$$

$$y_j = \begin{cases} 1 & \text{if } x_j = \text{TRUE} \\ 0 & \text{if } x_j = \text{FALSE} \end{cases}$$

e.g.  $C_i = x_1 \vee \bar{x}_2 \vee x_3$

$\Rightarrow$  constraint

$$y_1 + (1 - y_2) + y_3 \geq z_1$$



# LP for max-SAT

$$\max \sum_{i=1}^m z_i$$

$$\text{over } y \in [0,1]^n, z \in [0,1]^m$$

$$\text{s.t. } \sum_{j: x_j \in C_i} y_j + \sum_{j: \bar{x}_j \in C_i} (1 - y_j) \geq z_i \quad \text{for each clause } C_i$$

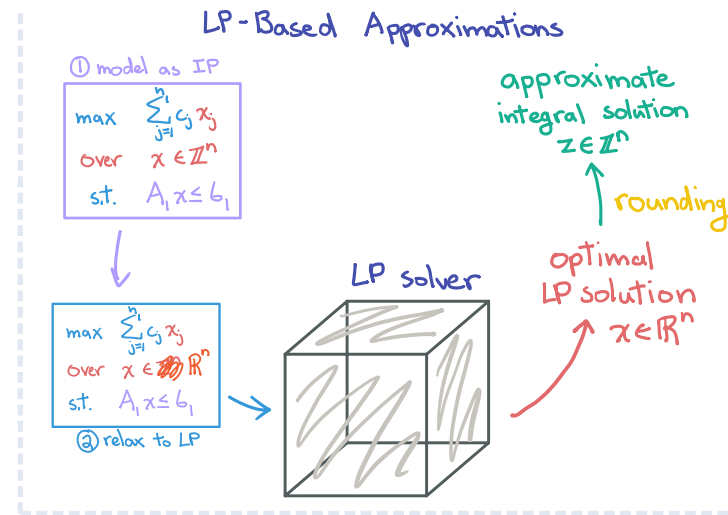
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$$y_j = \begin{cases} 1 & \text{if } x_j = \text{TRUE} \\ 0 & \text{if } x_j = \text{FALSE} \end{cases}$$

e.g.  $C_i = x_1 \vee \bar{x}_2 \vee x_3$

$\Rightarrow$  constraint

$$y_1 + (1 - y_2) + y_3 \geq z_1$$



IP for max-SAT

$$\max \sum_{i=1}^m z_i$$

over  $y \in \{0,1\}^n, z \in \{0,1\}^m$

$$\text{s.t. } \sum_{j: x_j \in C_i} y_j + \sum_{j: \bar{x}_j \in C_i} (1 - y_j) \geq z_i$$

NP-Hard

LP for max-SAT

$$\max \sum_{i=1}^m z_i$$

over  $y \in [0,1]^n, z \in [0,1]^m$

$$\text{s.t. } \sum_{j: x_j \in C_i} y_j + \sum_{j: \bar{x}_j \in C_i} (1 - y_j) \geq z_i$$

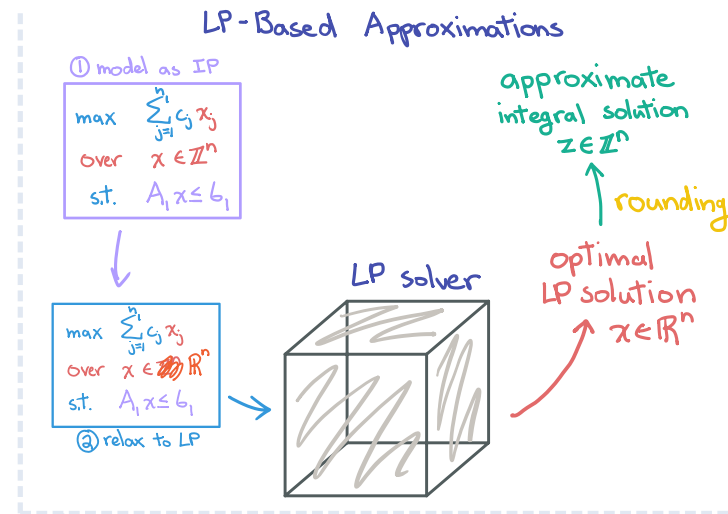
Solvable

$$\text{OPT}_{\text{IP}} \leq \text{OPT}_{\text{LP}}$$

# Rounding LP solution

$$y \in [0,1]^n \rightsquigarrow x \in \{\text{TRUE}, \text{FALSE}\}$$

ideas?



$$\begin{array}{ll} \max & \sum_{i=1}^m z_i \\ \text{over} & y \in [0,1]^n, z \in [0,1]^m \\ \text{s.t.} & \sum_{j: x_j \in C_i} y_j + \sum_{j: \bar{x}_j \in C_i} (1 - y_j) \geq z_i \\ & \text{for each clause } C_i \end{array}$$

$z_i = \begin{cases} 1 & \text{if } i\text{th clause satisfied} \\ 0 & \text{if not} \end{cases}$

$y_j = \begin{cases} 1 & \text{if } x_j = \text{TRUE} \\ 0 & \text{if } x_j = \text{FALSE} \end{cases}$

e.g., how to interpret  $y_i = \frac{1}{2}$ ?

# Rounding LP solution

$$y \in [0,1]^n \rightsquigarrow x \in \{\text{TRUE}, \text{FALSE}\}$$

For each variable  $x_j$ , independently, randomly set  $x_j = \text{t}$  with probability  $y_j$ , and  $x_j = \text{f}$  otherwise.

## LP-Based Approximations

① model as IP

$$\begin{array}{ll} \max & \sum_{j=1}^n c_j x_j \\ \text{over} & x \in \mathbb{Z}^n \\ \text{s.t.} & A_1 x \leq b_1 \end{array}$$

$$\begin{array}{ll} \max & \sum_{j=1}^n c_j x_j \\ \text{over} & x \in \mathbb{R}^n \\ \text{s.t.} & A_1 x \leq b_1 \end{array}$$

② relax to LP



approximate  
integral solution  
 $z \in \mathbb{Z}^n$

↑ rounding

optimal  
LP solution  
 $x \in \mathbb{R}^n$

$$\max \sum_{i=1}^m z_i$$

$$\text{over } y \in [0,1]^n, z \in [0,1]^m$$

$$\text{s.t. } \sum_{j: x_j \in C_i} y_j + \sum_{j: \bar{x}_j \in C_i} (1 - y_j) \geq z_i$$

for each clause  $C_i$

$$z_i = \begin{cases} 1 & \text{if } i\text{th clause satisfied} \\ 0 & \text{if not} \end{cases}$$

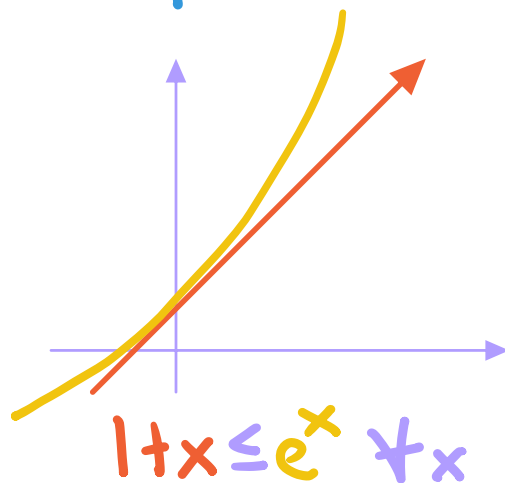
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claim: each clause  $C_i$  is satisfied w/ probability  $(1 - 1/e)z_i$

$P[C_i \text{ not satisfied}]$

$$\begin{aligned}
 &= \prod_{x_j \in C_i} (1 - y_j) \prod_{\bar{x}_j \in C_i} y_j \\
 &\leq \prod_{x_j \in C_i} e^{-y_j} \prod_{\bar{x}_j \in C_i} e^{-y_j} \\
 &= e^{-\sum_{x_j \in C_i} y_j - \sum_{\bar{x}_j \in C_i} y_j} = e^{-z_i}
 \end{aligned}$$



$$e^{-z_i} \leq (1-z_i)e^0 + z_i e^{-1} = 1 - (1 - 1/e)z_i$$

## LP-Based Approximations

① model as IP

$$\begin{aligned}
 &\max \sum_{j=1}^n c_j x_j \\
 &\text{over } x \in \mathbb{Z}^n \\
 &\text{s.t. } A_1 x \leq b_1
 \end{aligned}$$

$$\begin{aligned}
 &\max \sum_{j=1}^n c_j x_j \\
 &\text{over } x \in \mathbb{R}^n \\
 &\text{s.t. } A_1 x \leq b_1
 \end{aligned}$$

② relax to LP

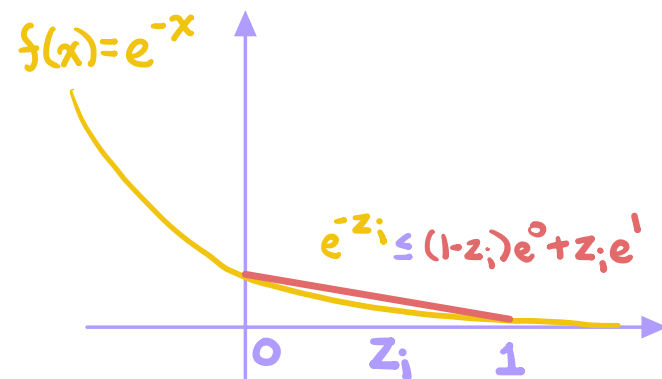


approximate  
integral solution  
 $z \in \mathbb{Z}^n$

rounding

optimal  
LP solution  
 $x \in \mathbb{R}^n$

$$\begin{aligned}
 &\max \sum_{i=1}^m z_i \\
 &\text{over } y \in [0,1]^n, z \in [0,1]^m \\
 &\text{s.t. } \sum_{j: x_j \in C_i} y_j + \sum_{j: \bar{x}_j \in C_i} (1 - y_j) \geq z_i \\
 &\quad \text{for each clause } C_i \\
 &z_i = \begin{cases} 1 & \text{if } i\text{th clause satisfied} \\ 0 & \text{if not} \end{cases} \\
 &y_j = \begin{cases} 1 & \text{if } x_j = \text{TRUE} \\ 0 & \text{if } x_j = \text{FALSE} \end{cases}
 \end{aligned}$$



For each variable  $x_j$ , independently, randomly set  $x_j = t$  with probability  $y_j$ , and  $x_j = f$  otherwise.

claim: each clause  $C_i$  is satisfied w/ probability  $(1 - 1/e) z_i$

claim:

$$E[\# \text{ sat. clauses}] \geq (1 - 1/e) \text{OPT}$$

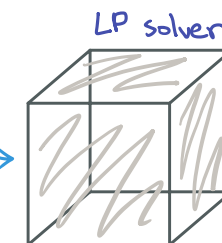
## LP-Based Approximations

① model as IP

$$\begin{array}{ll} \max & \sum_{j=1}^n c_j x_j \\ \text{over} & x \in \mathbb{Z}^n \\ \text{s.t.} & A_1 x \leq b_1 \end{array}$$

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② relax to LP



approximate  
integral solution  
 $z \in \mathbb{Z}^n$

rounding

optimal  
LP solution  
 $x \in \mathbb{R}^n$

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for each clause  $C_i$

$$z_i = \begin{cases} 1 & \text{if } i\text{th clause satisfied} \\ 0 & \text{if not} \end{cases}$$

$$y_j = \begin{cases} 1 & \text{if } x_j = \text{TRUE} \\ 0 & \text{if } x_j = \text{FALSE} \end{cases}$$

For each variable  $x_j$ , independently, randomly set  $x_j = \text{t}$  with probability  $y_j$ , and  $x_j = \text{f}$  otherwise.

$$E[\# \text{ sat. clauses}] \geq (1 - 1/e) \text{OPT}$$

Better?

# LP-Based Approximations

① model as IP

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③ relax to LP



approximate  
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 $z \in \mathbb{Z}^n$

↑ rounding

optimal  
LP solution  
 $x \in \mathbb{R}^n$

$$\max \sum_{i=1}^m z_i$$

$$\text{over } y \in [0,1]^n, z \in [0,1]^m$$

$$\text{s.t. } \sum_{j: x_j \in C_i} y_j + \sum_{j: \bar{x}_j \in C_i} (1 - y_j) \geq z_i$$

for each clause  $C_i$

$$z_i = \begin{cases} 1 & \text{if } i\text{th clause satisfied} \\ 0 & \text{if not} \end{cases}$$

$$y_j = \begin{cases} 1 & \text{if } x_j = \text{TRUE} \\ 0 & \text{if } x_j = \text{FALSE} \end{cases}$$



For each variable  $x_j$ , independently, randomly set  $x_j = t$  with probability  $y_j$ , and  $x_j = f$  otherwise.

claim: each clause  $C_i$  is satisfied  
w/ probability  $(1 - 1/e) z_i$

what if  $C_i$  has 1 variable?

$$P[\text{sat. } C_i] = z_i$$

~ 2 variables?

## LP-Based Approximations

① model as IP

$$\begin{array}{ll} \max & \sum_{j=1}^n c_j x_j \\ \text{over} & x \in \mathbb{Z}^n \\ \text{s.t.} & A_1 x \leq b_1 \end{array}$$

↓

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what if  $C_i$  has 1 variable?

$$P[\text{sat. } C_i] = z_i$$

~ 2 variables? e.g.  $C_i = x_1 \vee x_2$

$$P[C_i \text{ not sat}] = (1 - y_1)(1 - y_2)$$

AM-GM:

$$\leq \left( \frac{(1 - y_1) + (1 - y_2)}{2} \right)^2$$

$$ab \leq \left( \frac{a+b}{2} \right)^2$$

$$\leq \left( 1 - \frac{z_i}{2} \right)^2 \leq (1 - z_i) \left( 1 - \frac{0}{2} \right)^2 + z_i \left( 1 - \frac{1}{2} \right)^2$$

$$= 1 - z_i + z_i/4 = 1 - \frac{3}{4} z_i$$

## LP-Based Approximations

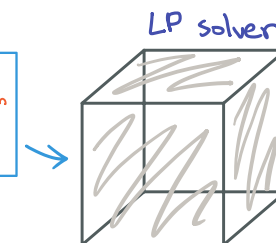
① model as IP

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② relax to LP



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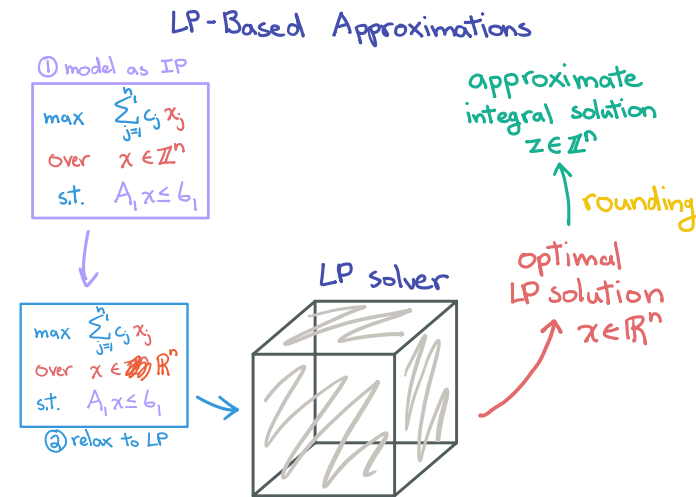
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$$z_i = \begin{cases} 1 & \text{if } i\text{th clause satisfied} \\ 0 & \text{if not} \end{cases}$$

$$y_j = \begin{cases} 1 & \text{if } x_j = \text{TRUE} \\ 0 & \text{if } x_j = \text{FALSE} \end{cases}$$

For each variable  $x_j$ , independently, randomly set  $x_j = \text{t}$  with probability  $y_j$ , and  $x_j = \text{f}$  otherwise.

$$E[\# \text{ sat. clauses}] \geq (1 - 1/e) \text{OPT}$$



| k        | oblivious   | LP             | average         |
|----------|-------------|----------------|-----------------|
| 1        | 1/2         | $z_i$          | $\geq (3/4)z_i$ |
| 2        | 3/4         | $(3/4)z_i$     | $\geq (3/4)z_i$ |
| 3        | 7/8         | $(1 - 1/e)z_i$ | $\geq (3/4)z_i$ |
| 4        | 15/16       | $(1 - 1/e)z_i$ | $\geq (3/4)z_i$ |
| 5        | 31/32       | $(1 - 1/e)z_i$ | $\geq (3/4)z_i$ |
| $\vdots$ |             |                |                 |
| k        | $1 - 1/2^k$ | $(1 - 1/e)z_i$ | $\geq (3/4)z_i$ |

$$\begin{array}{ll} \max & \sum_{i=1}^m z_i \\ \text{over} & y \in [0,1]^n, z \in [0,1]^m \\ \text{s.t.} & \sum_{j: x_j \in C_i} y_j + \sum_{j: \bar{x}_j \in C_i} (1 - y_j) \geq z_i \\ & \text{for each clause } C_i \end{array}$$

$z_i = \begin{cases} 1 & \text{if } i\text{th clause satisfied} \\ 0 & \text{if not} \end{cases}$

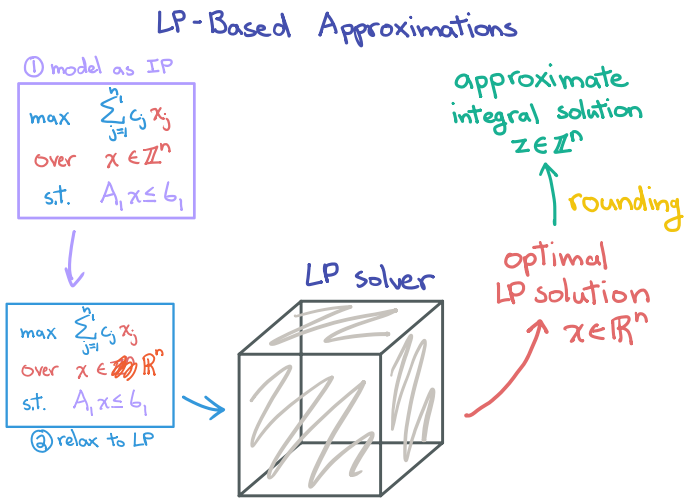
$y_j = \begin{cases} 1 & \text{if } x_j = \text{TRUE} \\ 0 & \text{if } x_j = \text{FALSE} \end{cases}$

Can we get  $\frac{3}{4}$ -APX?

- ① Flip a coin
- ② If heads, do oblivious rounding
- ③ If tails, do LP rounding

For each variable  $x_j$ , independently, randomly set  $x_j = t$  with probability  $y_j$ , and  $x_j = f$  otherwise.

claim: each clause  $C_i$  is satisfied  
w/ probability  $\frac{3}{4} z_i$



$$\begin{array}{ll} \max & \sum_{i=1}^m z_i \\ \text{over} & y \in [0,1]^n, z \in [0,1]^m \\ \text{s.t.} & \sum_{j: x_j \in C_i} y_j + \sum_{j: \bar{x}_j \in C_i} (1 - y_j) \geq z_i \end{array}$$

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$$z_i = \begin{cases} 1 & \text{if } i\text{th clause satisfied} \\ 0 & \text{if not} \end{cases}$$

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| k | oblivious   | LP             | average         |
|---|-------------|----------------|-----------------|
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| 3 | 7/8         | $(1 - 1/e)z_i$ | $\geq (3/4)z_i$ |
| 4 | 15/16       | $(1 - 1/e)z_i$ | $\geq (3/4)z_i$ |
| 5 | 31/32       | $(1 - 1/e)z_i$ | $\geq (3/4)z_i$ |
| ⋮ |             |                |                 |
| k | $1 - 1/2^k$ | $(1 - 1/e)z_i$ | $\geq (3/4)z_i$ |

today

Probability (& techniques)

Approximations via  
linear programming  
& randomized rounding

Algorithms

Max Sat

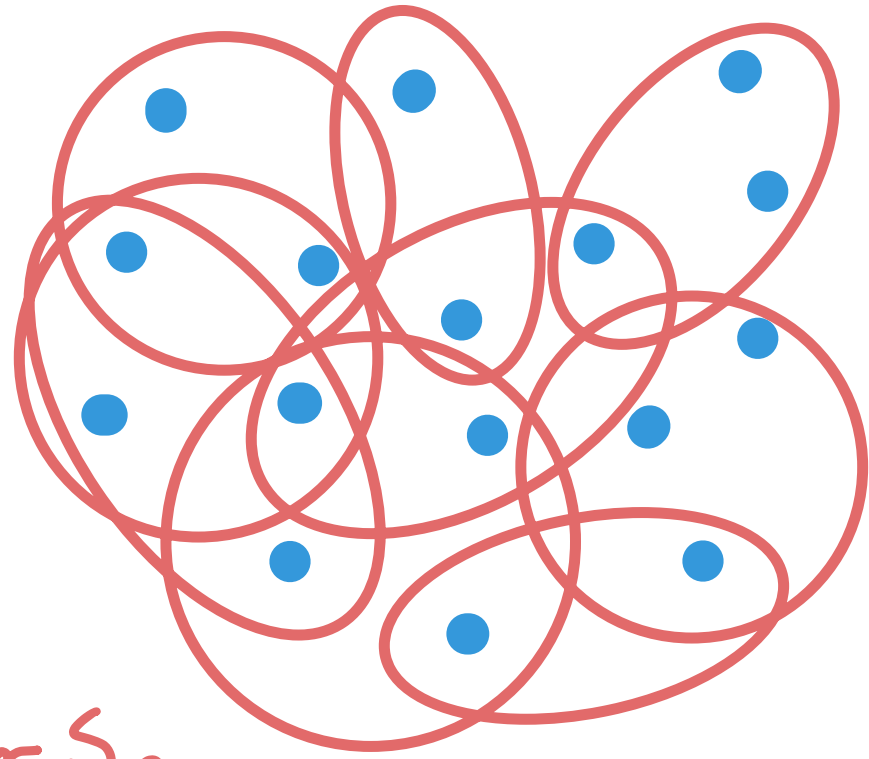
Set Cover

Covering  
Integer Programs

# Minimum Set Cover

$m$  points  $[m] \equiv \{1, \dots, m\}$

$n$  sets  $S_1, \dots, S_n \subseteq [m]$



Goal: select  $k$  sets  $S_{e_1}, S_{e_2}, \dots, S_{e_k}$   
minimizing  $k$ ,

$$\text{s.t. } S_{e_1} \cup S_{e_2} \cup \dots \cup S_{e_k} = [m]$$

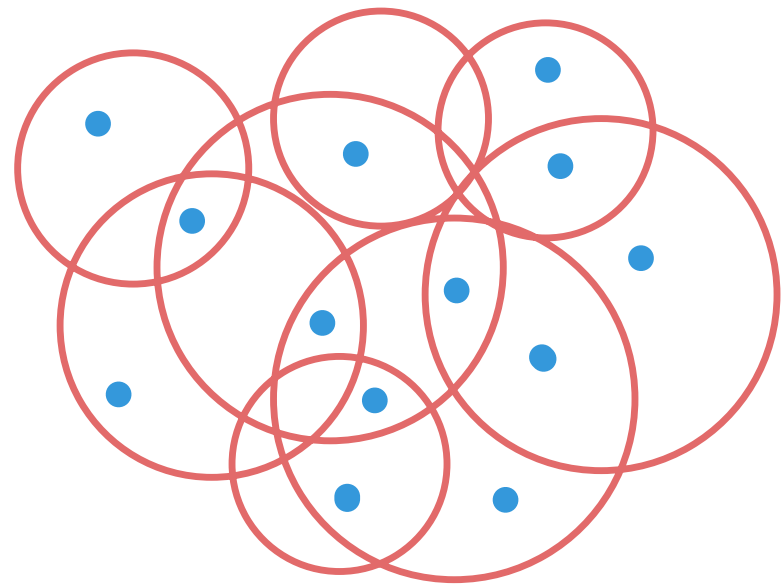
NP-Hard

$$\varphi(x_1, \dots, x_n) =$$

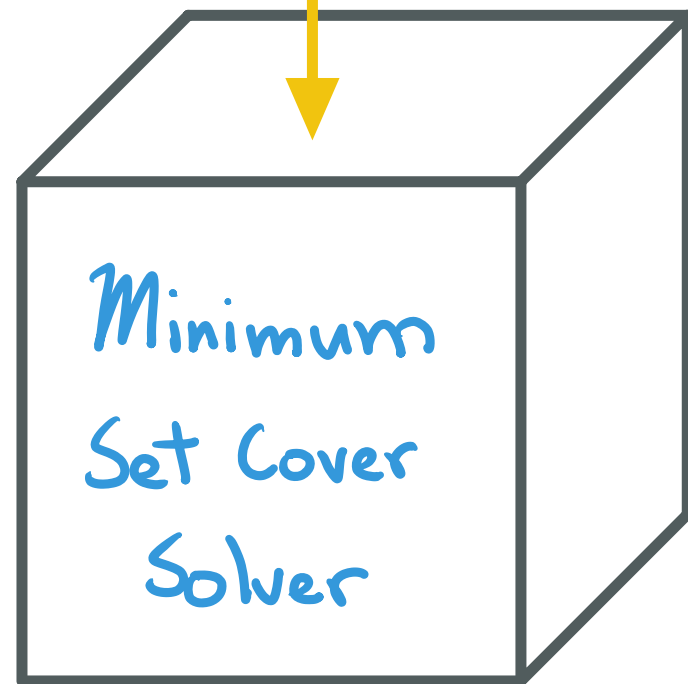
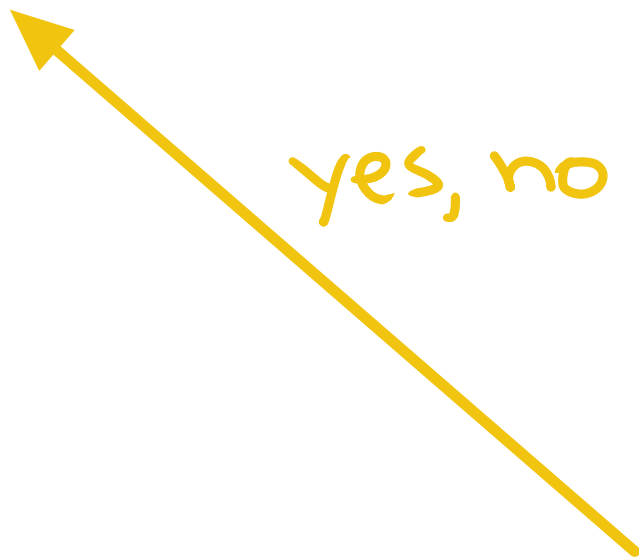
$$(x_1 \vee \bar{x}_2)$$

$$\wedge (x_2 \vee x_3 \vee \bar{x}_4)$$

$$\wedge m$$



yes, no

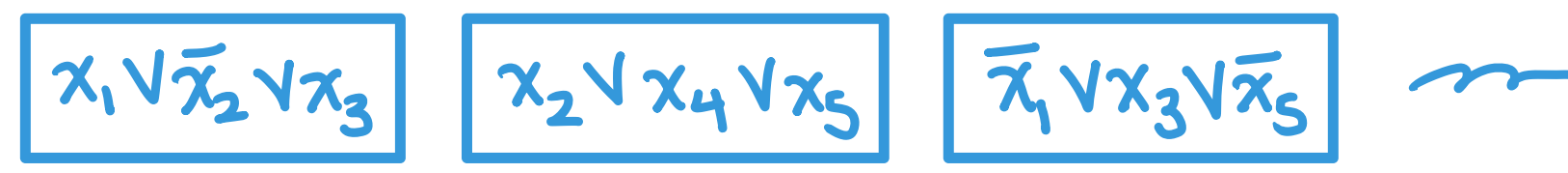


[pause here and try  
to think of a  
reduction from 3-SAT]

$$f(x) = (x_1 \vee \bar{x}_2 \vee x_3) \wedge (x_2 \vee x_4 \vee x_5) \wedge \dots$$

(n variables,  
m clauses)

elements  
per  
clause



sets =  
assignments



elements  
per variable



$f$  satisfiable  $\Leftrightarrow$  set cover of size n



# IP for set cover

$$\min \sum_{j=1}^n x_j$$

$$\text{over } x \in \{0, 1\}^n$$

$$\text{s.t. } \sum_{j: i \in S_j} x_j \geq 1 \quad \forall i \in [m]$$

$$x_j = \begin{cases} 1 & \text{if } S_j \text{ in set cover,} \\ 0 & \text{otherwise} \end{cases}$$

## LP-Based Approximations

① model as IP

$$\begin{array}{ll} \max & \sum_{j=1}^n c_j x_j \\ \text{over} & x \in \mathbb{Z}^n \\ \text{s.t.} & A_1 x \leq b_1 \end{array}$$



$$\begin{array}{ll} \max & \sum_{j=1}^n c_j x_j \\ \text{over} & x \in \mathbb{R}^n \\ \text{s.t.} & A_1 x \leq b_1 \end{array}$$

② relax to LP



optimal LP solution  $x \in \mathbb{R}^n$

rounding

approximate integral solution  $z \in \mathbb{Z}^n$

## Minimum Set Cover

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Goal: select  $k$  sets  $S_{e_1}, \dots, S_{e_k}$

minimizing  $k$ ,

s.t.  $S_{e_1} \cup \dots \cup S_{e_k} = [m]$

# LP for set cover

$$\min \sum_{j=1}^n x_j$$

$$\text{over } x \in [0,1]^n$$

$$\text{s.t. } \sum_{j: i \in S_j} x_j \geq 1 \quad \forall i \in [m]$$

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## LP-Based Approximations

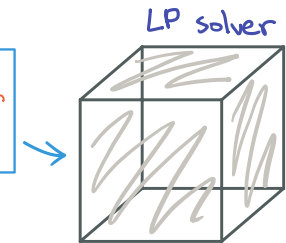
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approximate  
integral solution  
 $z \in \mathbb{Z}^n$

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IP for set cover

$$\min \sum_{j=1}^n x_j$$

over  $x \in \{0,1\}^n$

$$\text{s.t. } \sum_{j: i \in S_j} x_j \geq 1 \\ \forall i \in [m]$$

NP-Hard

LP for set cover

$$\min \sum_{j=1}^n x_j$$

over  $x \in [0,1]^n$

$$\text{s.t. } \sum_{j: i \in S_j} x_j \geq 1 \\ \forall i \in [m]$$

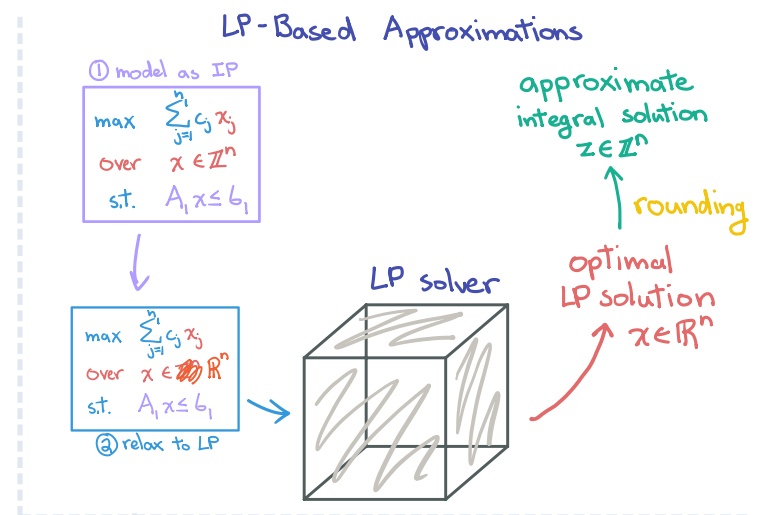
Polytime solvable

# Rounding LP solution

$$x \in [0,1]^n \rightsquigarrow F \subseteq \{S_1, \dots, S_n\}$$

Ideas?

How to interpret  $x_i = .5$ ?



## Minimum Set Cover

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minimizing  $k$ ,

$$\text{s.t. } S_{e_1} \cup \dots \cup S_{e_k} = [m]$$

$$\min \sum_{j=1}^n x_j$$

$$\text{over } x \in [0,1]^n$$

$$\text{s.t. } \sum_{j: i \in S_j} x_j \geq 1 \quad \forall i \in [m]$$

$$x_j = \begin{cases} 1 & \text{if } S_j \text{ in set cover,} \\ 0 & \text{otherwise} \end{cases}$$

# Rounding LP solution

$$x \in [0,1]^n \rightsquigarrow F \subseteq \{S_1, \dots, S_n\}$$

For each set  $S_j$ , independently, take  $S_j$  with probability  $x_j$ .

$$E[|F|] = \sum_j x_j = \text{OPT} \quad (\text{good!})$$

Fix a point  $i$ .

$$E[\text{sets in } F \text{ covering } i] = \sum_{j: i \in S_j} x_j \geq 1$$

$$P[F \text{ not cover } i] = \prod_{j: i \in S_j} (1 - x_j) \leq \prod_{j: i \in S_j} e^{-x_j}$$

$$= e^{-\sum_{j: i \in S_j} x_j} \leq e^{-1}$$

## LP-Based Approximations

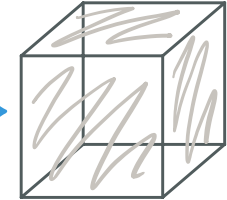
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② relax to LP



LP solver

approximate  
integral solution  
 $z \in \mathbb{Z}^n$

rounding

optimal  
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$$\text{s.t. } S_{e_1} \cup \dots \cup S_{e_k} = [m]$$

$$\min \sum_{j=1}^n x_j$$

over  $x \in [0,1]^n$

$$\text{s.t. } \sum_{j: i \in S_j} x_j \geq 1 \quad \forall i \in [m]$$

$$x_j = \begin{cases} 1 & \text{if } S_j \text{ in set cover,} \\ 0 & \text{otherwise} \end{cases}$$

# Rounding LP solution

$$x \in [0,1]^n \rightsquigarrow F \subseteq \{S_1, \dots, S_n\}$$

For each set  $S_j$ , independently, take  $S_j$  with probability  $x_j$ .

$$E[|F|] = \sum_j x_j = \text{OPT} \quad (\text{good!})$$

$$P[F \text{ not cover } i] = e^{-1}$$

so  $F$  covers each point w/  
constant probability

$\Rightarrow$   $F$  is a set cover w/ constant prob.

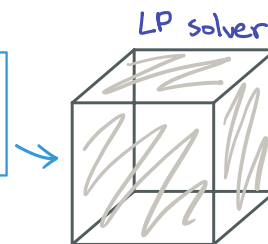
## LP-Based Approximations

① model as IP

$$\begin{array}{ll} \max & \sum_{j=1}^n c_j x_j \\ \text{over} & x \in \mathbb{Z}^n \\ \text{s.t.} & A_1 x \leq b_1 \end{array}$$

$$\begin{array}{ll} \max & \sum_{j=1}^n c_j x_j \\ \text{over} & x \in \mathbb{R}^n \\ \text{s.t.} & A_1 x \leq b_1 \end{array}$$

② relax to LP



approximate  
integral solution  
 $z \in \mathbb{Z}^n$

rounding

optimal  
LP solution  
 $x \in \mathbb{R}^n$

## Minimum Set Cover

$m$  points  $[m] \equiv \{1, \dots, m\}$

$n$  sets  $S_1, \dots, S_n \subseteq [m]$

Goal: select  $k$  sets  $S_{e_1}, \dots, S_{e_k}$   
minimizing  $k$ ,

$$\text{s.t. } S_{e_1} \cup \dots \cup S_{e_k} = [m]$$

$$\min \sum_{j=1}^n x_j$$

$$\text{over } x \in [0,1]^n$$

$$\text{s.t. } \sum_{j: i \in S_j} x_j \geq 1 \quad \forall i \in [m]$$

$$x_j = \begin{cases} 1 & \text{if } S_j \text{ in set cover,} \\ 0 & \text{otherwise} \end{cases}$$

# Rounding LP solution

$$x \in [0,1]^n \rightsquigarrow F \subseteq \{S_1, \dots, S_n\}$$

2. For each set  $S_j$  independently, take  $S_j$  with probability  $\min\{1, \alpha x_j\}$ .

for  $d > 1$  TBD.

$$E[|F|] = d \sum_j x_j = d \text{OPT}$$

$$P[F \text{ not cover } i] = \prod_{S_j \ni i} (1 - d x_j) \\ \leq e^{-d \sum_{S_j \ni i} x_j} \leq e^{-d}$$

$$P[F \text{ not set cover}] \leq \sum_i P[F \text{ not cover } i] \\ \leq m \cdot e^{-d} \leq \frac{1}{m^{\alpha d}} \text{ for } d = O(\log m)$$

## LP-Based Approximations

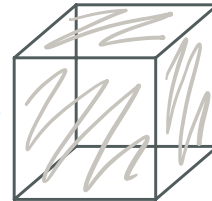
① model as IP

$$\begin{array}{ll} \max & \sum_{j=1}^n c_j x_j \\ \text{over} & x \in \mathbb{Z}^n \\ \text{s.t.} & A_j x \leq b_j \end{array}$$



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② relax to LP



LP solver

approximate  
integral solution  
 $z \in \mathbb{Z}^n$

rounding

optimal  
LP solution  
 $x \in \mathbb{R}^n$

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$$\min \sum_{j=1}^n x_j$$

over  $x \in [0,1]^n$

$$\text{s.t. } \sum_{j: i \in S_j} x_j \geq 1 \\ \forall i \in [m]$$

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for  $d > 1$  TBD.

$$E[|F|] = d \sum_j x_j = d \text{OPT}$$

$$P[F \text{ not set cover}] \leq \frac{1}{m^{\alpha d}}$$

$$\text{for } d = O(\log m)$$

$$\Rightarrow O(\log m)\text{-APX}$$

## LP-Based Approximations

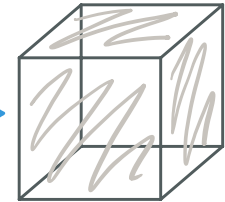
① model as IP

$$\begin{array}{ll} \max & \sum_{j=1}^n c_j x_j \\ \text{over} & x \in \mathbb{Z}^n \\ \text{s.t.} & A_j x \leq b_j \end{array}$$



$$\begin{array}{ll} \max & \sum_{j=1}^n c_j x_j \\ \text{over} & x \in \mathbb{R}^n \\ \text{s.t.} & A_j x \leq b_j \end{array}$$

② relax to LP



LP solver

approximate  
integral solution  
 $z \in \mathbb{Z}^n$

rounding

optimal  
LP solution  
 $x \in \mathbb{R}^n$

## Minimum Set Cover

$m$  points  $[m] \equiv \{1, \dots, m\}$

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$$\text{over } x \in [0,1]^n$$

$$\text{s.t. } \sum_{j: i \in S_j} x_j \geq 1 \quad \forall i \in [m]$$

$$x_j = \begin{cases} 1 & \text{if } S_j \text{ in set cover,} \\ 0 & \text{otherwise} \end{cases}$$



today

Probability (& techniques)

Approximations via  
linear programming  
& randomized rounding

Algorithms

Max Sat

Set Cover

Covering  
Integer Programs

# Covering IPs

$$\min \sum_j c_j x_j$$

$$\text{over } x \in \mathbb{Z}_{\geq 0}^n$$

$$\text{s.t. } \sum_{j=1}^n A_{ij} x_j \geq b_i \text{ for } i \in [m]$$

$$\text{where } A \in \mathbb{R}_{\geq 0}^{m \times n}, \quad b \in \mathbb{R}_{> 0}^m, \quad c \in \mathbb{R}_{> 0}^n$$

"demands"                      "costs"

WLOG  $A_{ij} \leq b_i \quad \forall i, j$  (why?)

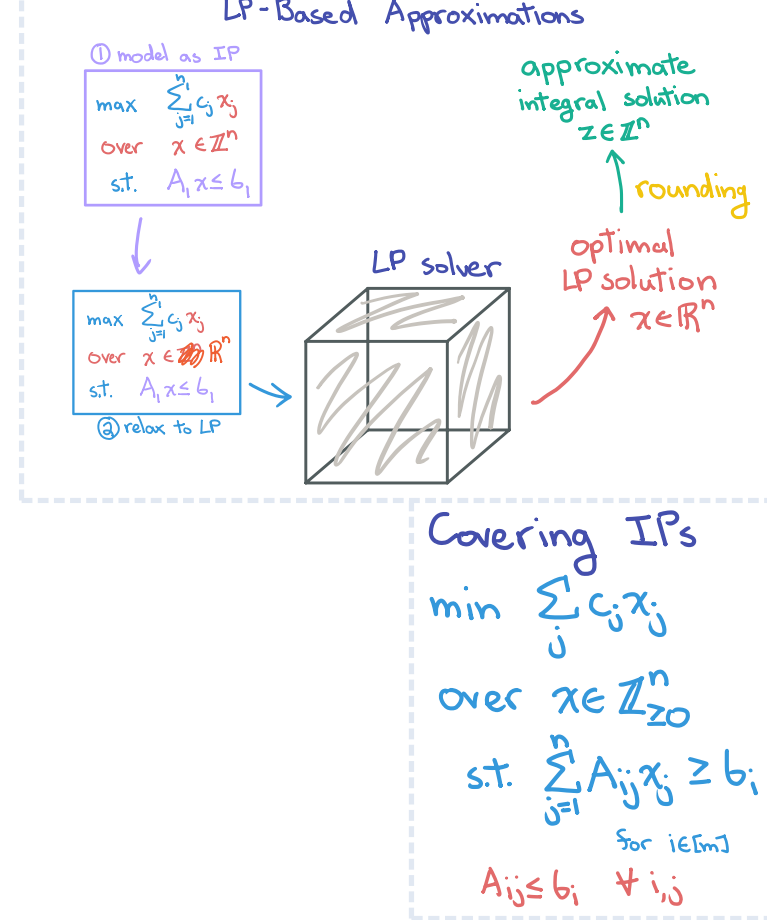
Set cover is special case w/  $A_{ij} \in \{0, 1\}$ ,  $b_i = 1$ ,  $c_j = 1$

# LP relaxation

$$\min \sum_j c_j x_j$$

$$\text{over } x \in \mathbb{Z}_{\geq 0}^n \quad \mathbb{R}_{\geq 0}^n$$

$$\text{s.t. } \sum_{j=1}^n A_{ij} x_j \geq b_i \quad \text{for } i \in [m]$$



# Rounding LP solution

$$x \in \mathbb{R}_{\geq 0}^n \rightsquigarrow z \in \mathbb{Z}_{\geq 0}^n$$

Ideas?

LP-Based Approximations

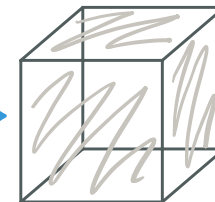
① model as IP

$$\begin{array}{ll} \max & \sum_{j=1}^n c_j x_j \\ \text{over} & x \in \mathbb{Z}^n \\ \text{s.t.} & A_1 x \leq b_1 \end{array}$$



$$\begin{array}{ll} \max & \sum_{j=1}^n c_j x_j \\ \text{over} & x \in \mathbb{R}^n \\ \text{s.t.} & A_1 x \leq b_1 \end{array}$$

② relax to LP



approximate  
integral solution  
 $z \in \mathbb{Z}^n$

rounding

optimal  
LP solution  
 $x \in \mathbb{R}^n$

LP relaxation

$$\begin{array}{ll} \min & \sum_j c_j x_j \\ \text{over} & x \in \mathbb{R}_{\geq 0}^n \\ \text{s.t.} & \sum_{j=1}^n A_{ij} x_j \geq b_i \\ & \text{for } i \in [m] \end{array}$$

Covering IPs

$$\begin{array}{ll} \min & \sum_j c_j x_j \\ \text{over} & x \in \mathbb{Z}_{\geq 0}^n \\ \text{s.t.} & \sum_{j=1}^n A_{ij} x_j \geq b_i \\ & \text{for } i \in [m] \\ & A_{ij} \leq b_i \quad \forall i, j \end{array}$$

# Rounding LP solution

$$x \in \mathbb{R}_{\geq 0}^n \rightsquigarrow z \in \mathbb{Z}_{\geq 0}^n$$

For each  $j \in [n]$ , independently, let

$$z_j = \begin{cases} \lceil \alpha x_j \rceil & \text{with probability } \alpha x_j - \lfloor \alpha x_j \rfloor \\ \lfloor \alpha x_j \rfloor & \text{otherwise.} \end{cases}$$

$$E[z_j] = \alpha x_j$$

$$E\left[\sum_j c_j z_j\right] = \alpha \sum_j c_j x_j = \alpha \text{OPT}$$

## LP-Based Approximations

① model as IP

$$\begin{array}{ll} \max & \sum_{j=1}^n c_j x_j \\ \text{over} & x \in \mathbb{Z}^n \\ \text{s.t.} & A_1 x \leq b_1 \end{array}$$

$$\begin{array}{ll} \max & \sum_{j=1}^n c_j x_j \\ \text{over} & x \in \mathbb{R}^n \\ \text{s.t.} & A_1 x \leq b_1 \end{array}$$

② relax to LP



approximate  
integral solution  
 $z \in \mathbb{Z}^n$

optimal  
LP solution  
 $x \in \mathbb{R}^n$

rounding

LP relaxation

$$\begin{array}{ll} \min & \sum_j c_j x_j \\ \text{over} & x \in \mathbb{R}_{\geq 0}^n \\ \text{s.t.} & \sum_{j=1}^n A_{ij} x_j \geq b_i \end{array}$$

for  $i \in [m]$

Covering IPs

$$\begin{array}{ll} \min & \sum_j c_j x_j \\ \text{over} & x \in \mathbb{Z}_{\geq 0}^n \\ \text{s.t.} & \sum_{j=1}^n A_{ij} x_j \geq b_i \end{array}$$

for  $i \in [m]$   
 $A_{ij} \leq b_i \quad \forall i, j$

For each  $j \in [n]$ , independently, let

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$$P[\sum_j A_{ij} z_j \leq b_i]$$

## LP-Based Approximations

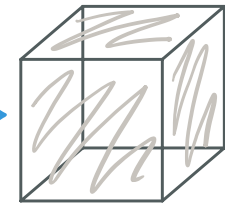
① model as IP

$$\begin{array}{ll} \max & \sum_{j=1}^n c_j x_j \\ \text{over} & x \in \mathbb{Z}^n \\ \text{s.t.} & A_1 x \leq b_1 \end{array}$$

↓

$$\begin{array}{ll} \max & \sum_{j=1}^n c_j x_j \\ \text{over} & x \in \mathbb{R}^n \\ \text{s.t.} & A_1 x \leq b_1 \end{array}$$

② relax to LP



LP solver

optimal LP solution  $x \in \mathbb{R}^n$

rounding

approximate integral solution  $z \in \mathbb{Z}^n$

LP relaxation

$$\begin{array}{ll} \min & \sum_j c_j x_j \\ \text{over} & x \in \mathbb{R}_{\geq 0}^n \\ \text{s.t.} & \sum_{j=1}^n A_{ij} x_j \geq b_i \\ & \text{for } i \in [m] \end{array}$$

Covering IPs

$$\begin{array}{ll} \min & \sum_j c_j x_j \\ \text{over} & x \in \mathbb{Z}_{\geq 0}^n \\ \text{s.t.} & \sum_{j=1}^n A_{ij} x_j \geq b_i \\ & \text{for } i \in [m] \\ & A_{ij} \leq b_i \quad \forall i, j \end{array}$$

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## LP-Based Approximations

① model as IP

$$\begin{aligned} \max \quad & \sum_{j=1}^n c_j x_j \\ \text{over } & x \in \mathbb{Z}^n \\ \text{s.t. } & A_1 x \leq b_1 \end{aligned}$$

$$\begin{aligned} \max \quad & \sum_{j=1}^n c_j x_j \\ \text{over } & x \in \mathbb{R}^n \\ \text{s.t. } & A_1 x \leq b_1 \end{aligned}$$

② relax to LP



approximate  
integral solution  
 $z \in \mathbb{Z}^n$

rounding  
optimal  
LP solution  
 $x \in \mathbb{R}^n$

LP relaxation

$$\begin{aligned} \min \quad & \sum_j c_j x_j \\ \text{over } & x \in \mathbb{R}_{\geq 0}^n \\ \text{s.t. } & \sum_{j=1}^n A_{ij} x_j \geq b_i \\ & \text{for } i \in [m] \end{aligned}$$

Covering IPs

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Multiplicative Chernoff bound

let  $X_1, \dots, X_n \in [0, 1]$  be independent

$$\mu = E[X_1 + \dots + X_n] \in (0, 1)$$

$$P[X_1 + \dots + X_n \leq (1 - \epsilon)\mu] \leq e^{-\epsilon^2 \mu / 2}$$

For each  $j \in [n]$ , independently, let

$$z_j = \begin{cases} \lceil \alpha x_j \rceil & \text{with probability } \alpha x_j - \lfloor \alpha x_j \rfloor \\ \lfloor \alpha x_j \rfloor & \text{otherwise.} \end{cases}$$

## LP-Based Approximations

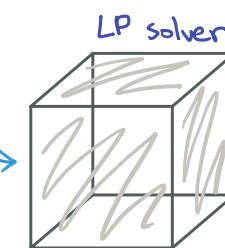
① model as IP

$$\begin{array}{ll} \max & \sum_{j=1}^n c_j x_j \\ \text{over} & x \in \mathbb{Z}^n \\ \text{s.t.} & A_1 x \leq b_1 \end{array}$$

↓

$$\begin{array}{ll} \max & \sum_{j=1}^n c_j x_j \\ \text{over} & x \in \mathbb{R}^n \\ \text{s.t.} & A_1 x \leq b_1 \end{array}$$

② relax to LP



approximate  
integral solution  
 $z \in \mathbb{Z}^n$

↑ rounding

optimal  
LP solution  
 $x \in \mathbb{R}^n$

LP relaxation

$$\begin{array}{ll} \min & \sum_j c_j x_j \\ \text{over} & x \in \mathbb{R}_{\geq 0}^n \\ \text{s.t.} & \sum_{j=1}^n A_{ij} x_j \geq b_i \\ & \text{for } i \in [m] \end{array}$$

Covering IPs

$$\begin{array}{ll} \min & \sum_j c_j x_j \\ \text{over} & x \in \mathbb{Z}_{\geq 0}^n \\ \text{s.t.} & \sum_{j=1}^n A_{ij} x_j \geq b_i \\ & \text{for } i \in [m] \\ & A_{ij} \leq b_i \quad \forall i, j \end{array}$$

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$$\mu = \mathbb{E}[X_1 + \dots + X_n] \in (0, 1)$$

$$P[X_1 + \dots + X_n \leq (1 - \epsilon)\mu] \leq e^{-\epsilon^2 \mu / 2}$$



# Multiplicative Chernoff bound

let  $X_1, \dots, X_n \in [0, 1]$  be independent

$$\mu = E[X_1 + \dots + X_n]$$

$$\epsilon \in (0, 1)$$

$$\bullet \quad P[X_1 + \dots + X_n \geq (1 + \epsilon)\mu] \leq e^{-\epsilon^2 \mu / 3}$$

$$\bullet \quad P[X_1 + \dots + X_n \leq (1 - \epsilon)\mu] \leq e^{-\epsilon^2 \mu / 2}$$

(proof at end if time permits)

